Lecture *Hierarchical Planning*

Chapter: *Complexity Results for Plan Existence*

Dr. Pascal Bercher

Institute of Artificial Intelligence,
Ulm University, Germany

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Overview:

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- The *plan verification problem*:
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- Last, but *not-at-all least*: they help understanding the problem! (Understanding the problem should always be the first step.)

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- A problem is called *undecidable* if it is not decidable.

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- Note: semi-decidable problems (sets) are also called, among others, *recursively enumerable*.

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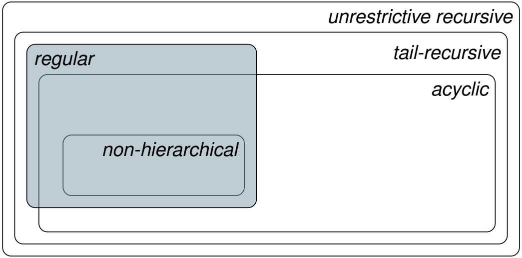

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- In case $\mathcal{P}$ uses an *initial task network* tn_I rather than an *initial task* c_I, then tn_I needs to be totally ordered as well.

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- A planning problem is called regular if all methods are regular.

Note: In case the planning problem features an initial task network, a problem is defined as regular if this network is regular, too. (Although this restriction in not necessary with regard to the results that base upon it.)

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Formally, the restrictions on recursion are defined in terms of so-called *stratifications*.

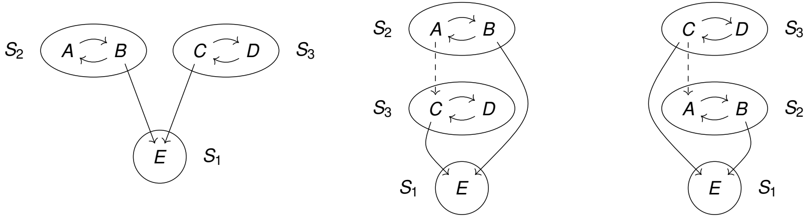
Stratifications

A stratification is defined as follows:

- A set $\leq \subseteq C \times C$ is called a *stratification* if it is a total preorder (i.e., reflexive, transitive, and *total*)

Stratifications: Example

(Non-)Examples for Stratifications:



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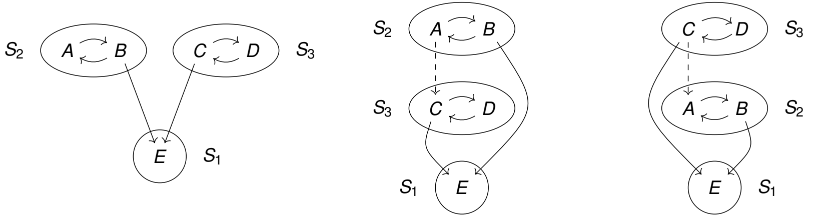

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- $\leq_a = \{(A, B), (B, A), (C, D), (D, C), (E, B), (E, C)\}$
- $\leq_a$ is not a stratification, as it is not total

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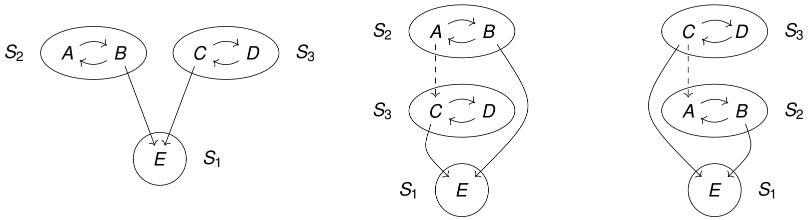

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- The *height of a stratification* is the number of its strata.

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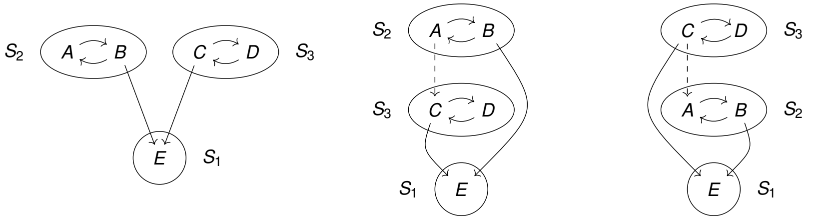

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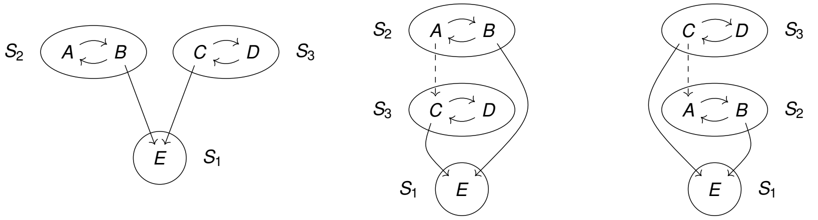

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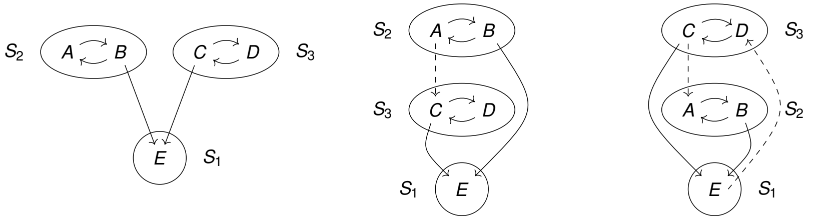

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- $\leq_b$ and $\leq_c$ have a height of 3.
- If we add, e.g., an edge from E to D in $\leq_c$, i.e., the tuple (D, E), then we only have *a single* stratification with height 1.

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An HTN problem $\mathcal{P}$ is called *tail-recursive* if there is a stratification $\leq$ on the compound tasks C of $\mathcal{P}$ with the following property:

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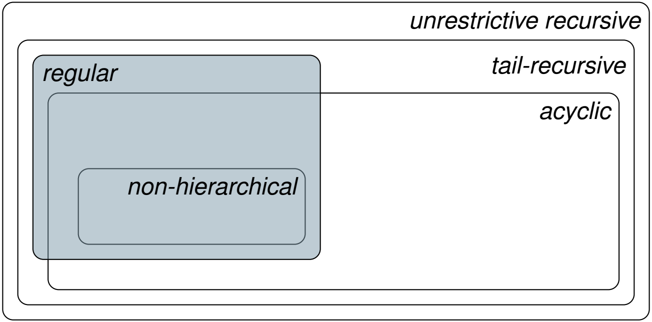
Overview of Problem Classes



Notes:

- Do not confuse these *problem classes* with the *language classes*!
- Totally ordered problems are not shown because this restriction is independent of all the ones depicted.

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  - (This reduces the hardness of the plan existence problem of length $2n$ to two problems of length n each.)

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  - For all states s_1, s_2, we want to know whether there is a plan from s_1 to s_2. This is done via asking:
  - Is there a plan of length $\leq n$ from s_1 to s' and another from s' to s_2?
  - (This reduces the hardness of the plan existence problem of length $2n$ to two problems of length n each.)
  - By iterating over all states, this requires polynomial space.

Complexity of the General Case, Hardness

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- Number of operators is proportional to number of transitions times tape squares.

More Complexity Results

There are several further cases that can be studied, e.g.:

- Take the number of preconditions/effects into account (special cases are often revealed via looking into the reductions).

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- Perform *partial* relaxations by ignoring only some parts (e.g., delete effects) of the model.
- Take dependencies between actions into account (they can be represented as graphs, the properties of which can be exploited).

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- Translate the production rules to decomposition methods. That way only words in $L(G)$ and $L(G')$ can be produced.
- Any solution *tn* contains the word ω – encoded as action sequence – twice: once produced by G and once produced by G'. The action encodings ensure that no other task networks are executable.

Undecidability Proof, cont'd – by Example

Proof idea by example:

Let $G = (\overbrace{\Gamma = \{H, Q\}}^{\text{non-terminal symbols}}, \overbrace{\Sigma = \{a, b\}}^{\text{terminal symbols}}, \overbrace{R}^{\text{production rules}}, \overbrace{H}^{\text{start symbol}})$
and $G' = (\Gamma' = \{D, F\}, \Sigma' = \{a, b\}, R', D)$ be formal grammars.

Production rules R: $H \mapsto aQb$ $\quad Q \mapsto aQ \mid bQ \mid a \mid b$

Production rules R': $D \mapsto aFD \mid ab$ $\quad F \mapsto a \mid b$

Undecidability Proof, cont'd – by Example

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Contructed HTN problem with desired solution set:

$$\mathcal{P} = (V, \overbrace{\{H, Q, D, F\}}^{C}, \overbrace{\{a, b, a', b'\}}^{P}, \delta, M, \overbrace{\{v_{turn:G}\}}^{\text{initial state}}, tn_I, \overbrace{\{v_{turn:G}\}}^{\text{goal description}})$$

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$M = M(G) \cup M(G')$ (translated production rules of G' and G')

$$tn_I = (\underbrace{\{t, t'\}}_{T}, \underbrace{\emptyset}_{\prec}, \underbrace{\{(t, H), (t', D)\}}_{\alpha})$$

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- However, for HTN planning: no! In case there is a solution we can prove this eventually (we just never know when, i.e., whether this is still going to happen).
- In other words: HTN planning is also(!) semi-decidable.
  undecidable + semi-decidable is also called *strictly semi-decidable*.

Semi-decidability Proof, cont'd

Theorem

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Proof:

Reminder: We need to find a function $\chi_N : M \to \{undef, 1\}$ with:

$$\chi_N(n) = \begin{cases} 1 & \text{if } n \in N \\ undef & \text{otherwise} \end{cases}$$

(Here, M is the set of all HTN planning problems. N is its subset of problems with a solution.)

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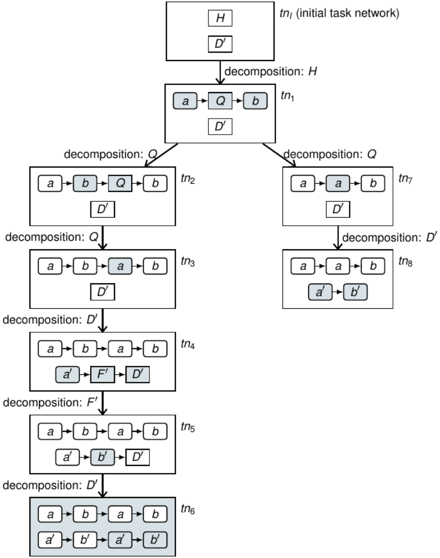
(Here, M is the set of all HTN planning problems. N is its subset of problems with a solution.)

Let $n = \mathcal{P}$. Define χ_N as a BFS procedure (starting with the initial task network) that returns 1 if and only if it discovered a solution to $\mathcal{P}$ (we can also return *undef* in case it can prove it to be unsolvable).

Influence of Task Insertion



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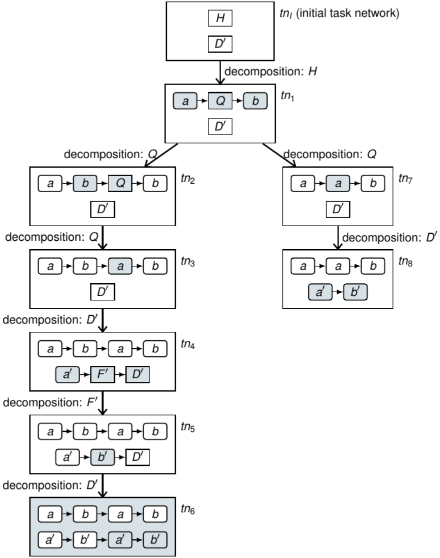

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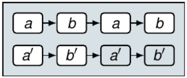
Task network tn_6 is a solution!



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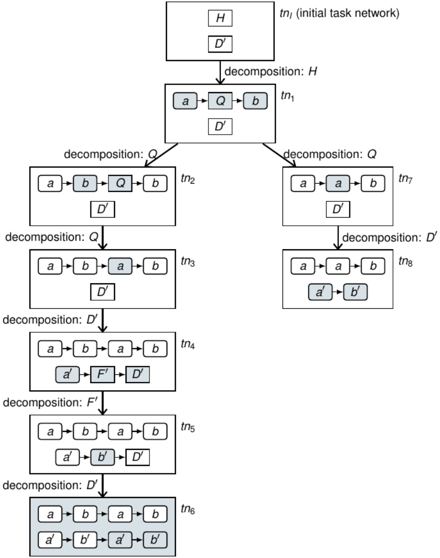

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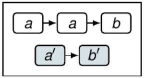
Task network tn_8 is no solution!



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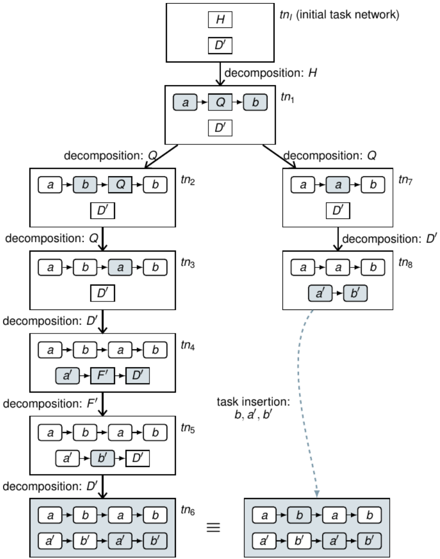

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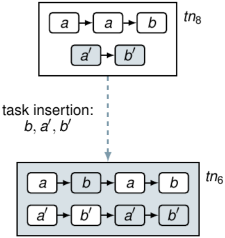

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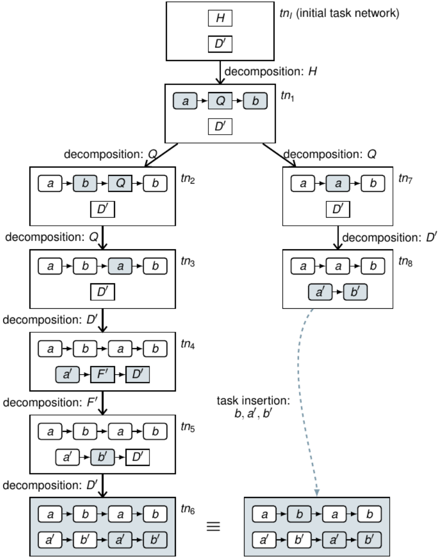

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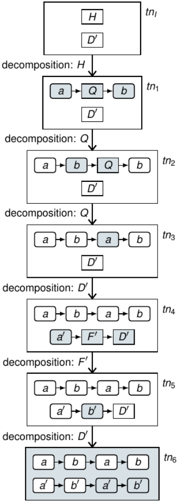
Observation:

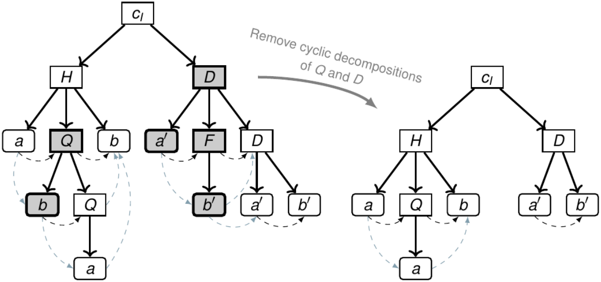
In TIHTN planning, recursion is not required.

Complexity of TIHTN Planning (Membership)



Theorem: TIHTN planning is in **NEXPTIME**

Idea: Restrict to *acyclic* decompositions, fill the rest with task insertion, and verify.



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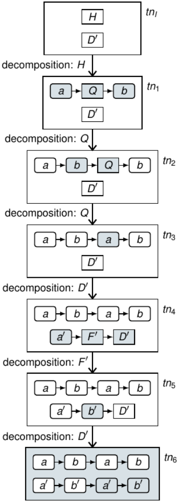

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1. Step: Guess an acyclic decomposition:

The guessed decomposition tree describes at most $b^{|C|+1}$ decompositions.

(C = set of compound tasks)
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Verify in $O(b^{|C|+1})$ whether the tree describes a correct sequence of decompositions.

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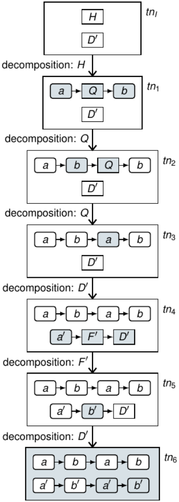

Theorem: TIHTN planning is in **NEXPTIME**

2. Step: Guess the actions and orderings to be inserted.

The (guessed) decomposition tree results into a task network with at most $\leq b^{|C|+1}$ tasks.

Between each two actions, at most $2^{|V|}$ actions need to be inserted to achieve the next precondition.

($|V|$ = number of state variables)

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- The proof is not provided in this lecture.

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- None of the restrictions of the hierarchy matters for TIHTN problems.
- TIHTN problems are less expressive than HTN problems (also cf. language results).

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The previous reduction already used a totally ordered TIHTN problem.

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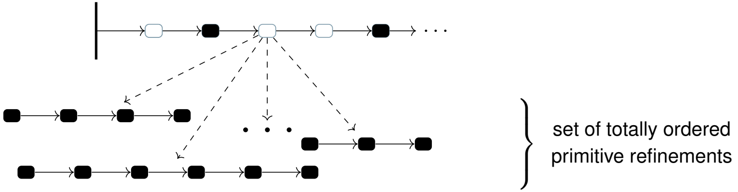
- Since plans are totally ordered, the only means of choosing the right refinement for a given compound task is to produce a suitable successor state.



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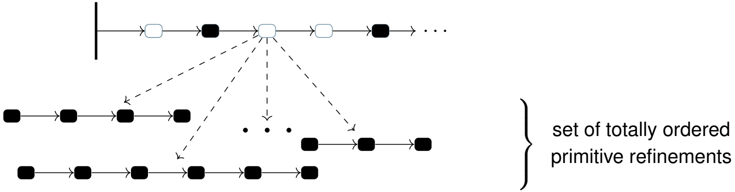

- There are only finitely many states that can be produced by the refinements of a given compound task.

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  - Continue as long as at least one value ? is changed.

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- Verify in $O(b^{|C|+1})$ whether the decompositions can be applied in sequence.

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Deciding whether an acyclic HTN planning problem has a solution is $\mathbb{NEXPTIME}$*-complete*.

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Do the same as for TIHTN problems, but without the task insertion part:

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- That way, the size of any task network is bounded by the size of the largest task network in the model.

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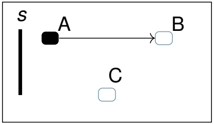

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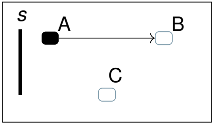

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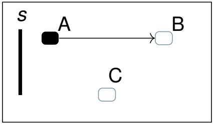

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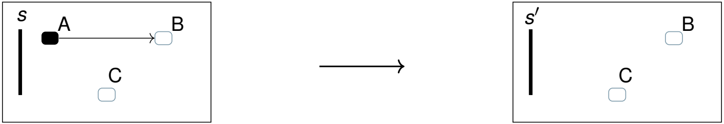

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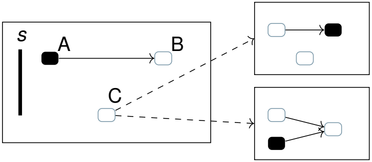

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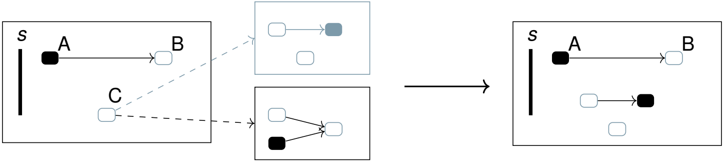

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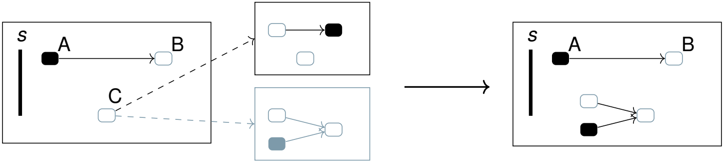

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- The initial task is X.

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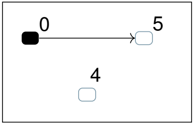
Consider the following initial task network of size 3:



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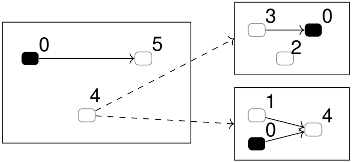

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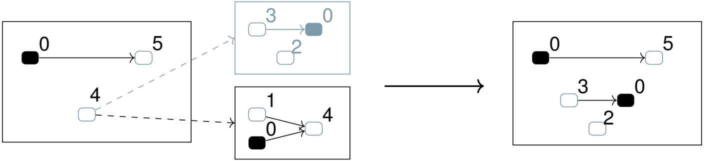

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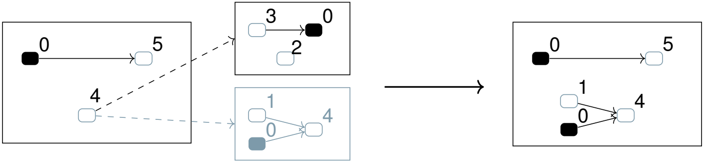

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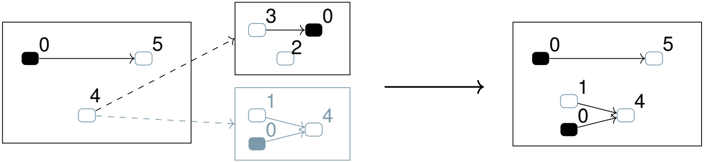

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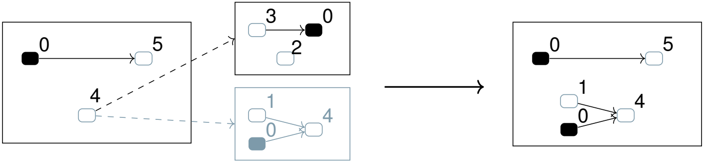

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