Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Sequents, Semantics, and Propositional Natural Deduction — Conjunction, Implication, Theorems

Pascal Bercher

AI Group
School of Computing
College of Engineering and Computer Science
the Australian National University

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Australian National University

Introduction

Recap: Connectives and Formulae: Syntax

The main connective dictates the type of a formula:

- if main connective is $\neg$, formula is a *negation*
- ... $\wedge$, ... *conjunction*
- ... $\vee$, ... *disjunction*
- ... $\rightarrow$, ... *implication*
- ... $\leftrightarrow$, ... *double-implication*

Recap: Connectives and Formulae: Semantics

What do these connectives *mean*?

- The "intended meaning" of connectives is expressed by truth tables:

p	$\neg$
0	1
1	0

p	q	$\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$\vee$
0	0	0
0	1	1
1	0	1
1	1	1

p	q	$\rightarrow$
0	0	1
0	1	1
1	0	0
1	1	1

- The truth value of a formula ϕ is defined by evaluating the formula under a given *interpretation*, which is an assignment of all propositional symbols.

Sequents and Natural Deduction: What and Why?: Sequents

- We want to know when one logical formula follows logically from another.
- Suppose we know that "p is true', e.g., due to some observation (technically: thus know that it is *interpreted* as true), and we know that $p \to q$ holds as well. Then we can logically conclude that q also holds!
- We can express this with *sequents*: $p, p \to q \models q$

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- We can express this with *sequents*: $p, p \to q \models q$
- These conclusions can be arbitrarily complicated, however! I.e., it might not be obvious that the conclusion follows from the premises.
- We use Natural Deduction to "manipulate sequents" step-wise thus "showing" validity.

Sequents

Introduction

Our convention:

- Letters from the *end* of the alphabet: set
- ... *beginning* ... : single object of the kind that's in the set

This represents a *valid* sequent: $X \models A$

- Read it: Formula A follows (logically) from the formulae in X
- For example, "q follows from p and $p \rightarrow q$"
- We write down: $p, p \rightarrow q \models q$
  but that's just short for: $\underbrace{\{p, p \rightarrow q\}}_{X} \models \underbrace{q}_{A}$
- Also $X, Y \models A$ is short for $X \cup Y \models A$,
  $X, B \models A$ is short for $X \cup \{B\} \models A$, and
  $X, B, C \models A$ is short for $X \cup \{B, C\} \models A$.

Another Example for a Valid Sequent

Previous example: $p, p \to q \models q$

But what if the conclusion isn't a "true" proposition (i.e., that's interpreted by 1)? What if it's a formula? What would that mean?

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What does it mean for a *formula* to follow logically?

- Assume we know $a \land (b \lor c)$ "holds", does $(b \lor c)$ follow as well?
- What does this even *mean*? We don't have the *property* "hold"?!

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What does it mean for a *formula* to follow logically?

- Assume we know $a \wedge (b \vee c)$ "holds", does $(b \vee c)$ follow as well?
- What does this even *mean*? We don't have the *property* "hold"?!

Answers:

- Yes, $a \wedge (b \vee c) \models (b \vee c)$ holds, i.e., it's a valid sequent!
- The formal definition is based on interpretations.

Semantically Valid Sequents

Definition:
$X \models A$ means the sequent is *valid*. This is the case if and only if:

- *A is true for every interpretation for which all the formulae in X are true.* Or, equivalently:
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How to check/test/prove $X \models A$? Create the proof tables!

- Create a table t_X for all formulae in X (all need to be true)
- Create another table t_A for A and check validity criterion.

Checking Validity, Example 1

How to prove $a \wedge (b \vee c) \models (b \vee c)$? *done live*

Checking Validity, Example 2

Show $\overbrace{(p \lor q) \to r, p}^{X} \models \overbrace{(p \to r) \land (q \to r)}^{A}$

Table t_X for premises:

p	q	r	$p \lor q$	$(p \lor q) \to r$	X
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Recall the definition: The sequent is valid if all interpretations that make X true also make A true!

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Table t_A for conclusion:

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Only two interpretations exist that make all $x \in X$ true:

1. $I_1(p) = I_1(r) = 1, I_1(q) = 0$
2. $I_2(p) = I_2(q) = I_2(r) = 1$

Both of them make A true! Thus, $X \models A$.

Natural Deduction

Motivation

So, it's all about finding out finding out whether some formula follows logically from the interaction of many others!

- E.g. you might have a *huge* knowledge base *KB* of rules. Maybe a medical database with (certified, based on experience or research) rules stating which symptoms indicate diseases or affected organs etc.
- What if we have a hypothesis about another rule that's not yet in the system?
  E.g., *if symptom p is present, it cannot be disease q.* If that's true, it would mean $KB \models p \rightarrow \neg q$.

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So we can just use truth tables and we are done, right?

Well, in theory, yes. But ... efficiency!

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  - We start with Natural Deduction!

Natural Deduction and Derivations

- Natural deduction is pure syntax manipulation and acts as *proof system*.
- Natural Deduction exploits *derivations*.
- A derivation is a finite sequence of formulae, which are derived from each other based on elementary formula manipulations (“1-step inference rules”)
- For each connective we will use two rules: one for *introducing* it, and one for *eliminating* it.

Syntax of Sequents

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- So, if some sequent $X \vdash A$ is given, we are interested in finding out whether it is actually *valid*, denoted by $X \models A$.
- To show that it's valid, we use Natural deduction.
- Only in the second part you will formally learn the relationship between these two concepts $\vdash$ and $\models$. I.e., you will learn how to show that each sequent $X \vdash A$ that is proved by some proof system (like Natural Deduction) is actually valid.

Conjunction

The 1-Step Rules: And-Elimination

What are the 1-step rules for dealing with conjunction?

Elimination rule:

$$\frac{A \wedge B}{A} \wedge E \qquad\qquad \frac{A \wedge B}{B} \wedge E$$

Which reads: If we derived $A \wedge B$, we can derive both A and B.

The 1-Step Rules: And-Introduction

What are the 1-step rules for dealing with conjunction?

Introduction rule:

$$\frac{A \qquad B}{A \wedge B} \wedge I$$

Which reads: If we derived A and we derived B, we can derive $A \wedge B$.

Proof Syntax / Notation: Overview

- How to write down proofs?
- There are many different notations that describe the same thing
- We introduce two:
  - Tree-like representation of the applied rules (just since it's another standard)
  - list-like representation (only use that one!)

Proof Syntax / Notation: Tree- and List-like Representations

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Proof Syntax / Notation: Tree- and List-like Representations

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- Sequence of derivations: $\underbrace{p \wedge q}_{\text{premise}}, \underbrace{q}_{\wedge E}, \underbrace{p}_{\wedge E}, \underbrace{q \wedge p}_{\wedge I \text{ and conclusion}}$
- In the tree-like format:

$$\dfrac{\dfrac{p \wedge q}{q}\wedge E \qquad \dfrac{p \wedge q}{p}\wedge E}{q \wedge p}\wedge I$$

- Leaves are assumptions, root is conclusion
- Advantages: Makes the proof structure obvious
- In exercises, etc: *Do not use it, unless we ask you to!*

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column 1: assumption number *column 2:* line number
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α_1	(1)	$p \wedge q$	A
α_1	(2)	q	1 $\wedge E$

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α_1	(3)	p	1 $\wedge E$	$\equiv$	$p \wedge q \vdash p$
α_1	(4)	$q \wedge p$	2,3 $\wedge I$	$\equiv$	$p \wedge q \vdash q \wedge p$

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- **Note:** Each line represents a sequent! (Sequence of sequents.)

The 1-Step Rules (Based on Sequents): Derivation Rules

- Derivation Rules as considered so far:

$$\frac{A \wedge B}{A}\wedge E \qquad \frac{A \wedge B}{B}\wedge E \qquad \frac{A \quad\quad B}{A \wedge B}\wedge I$$

The 1-Step Rules (Based on Sequents): Derivation Rules

- Derivation Rules as considered so far:

$$\frac{A \land B}{A}\land E \qquad \frac{A \land B}{B}\land E \qquad \frac{A \qquad B}{A \land B}\land I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \land B}{X \vdash A}\land E \qquad \frac{X \vdash A \land B}{X \vdash B}\land E$$

The 1-Step Rules (Based on Sequents): Derivation Rules

- Derivation Rules as considered so far:

$$\frac{A \wedge B}{A}\wedge E \qquad \frac{A \wedge B}{B}\wedge E \qquad \frac{A \quad\quad B}{A \wedge B}\wedge I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \wedge B}{X \vdash A}\wedge E \qquad \frac{X \vdash A \wedge B}{X \vdash B}\wedge E \qquad \frac{X \vdash A \quad\quad Y \vdash B}{X, Y \vdash A \wedge B}\wedge I$$

$\rightarrow$ I.e., now we see how premises accumulate!

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$$p, q \vdash p \wedge q$$

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$p, q \vdash p \wedge q$

α_1	(1)	p	A

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$p, q \vdash p \wedge q$

α_1	(1)	p	A
α_2	(2)	q	A

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$p, q \vdash p \wedge q$

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$p, q \vdash p \wedge q$

α_1	(1)	p	A	$\equiv$	$p \vdash p$	(by assumption)
α_2	(2)	q	A	$\equiv$	$q \vdash q$	(by assumption)
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$	$\equiv$	$p, q \vdash p \wedge q$	$(\wedge I)$

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \wedge I$$

Implication

Introduction

- Now we consider the "if . . . , then . . . " connective: implication!
- E.g.,
  - $p \rightarrow q$: "if it is raining (p), then the ground is wet (q)"
  - Here, p is the *antecedent* and q the *consequent*

Introduction

- Now we consider the "if . . . , then . . . " connective: implication!
- E.g.,
  - $p \rightarrow q$: "if it is raining (p), then the ground is wet (q)"
  - Here, p is the *antecedent* and q the *consequent*
  - $(p \wedge q) \rightarrow r$:
    - All tigers are carnivores (p) } premises
    - Timmy is a tiger (q) } premises
    - Thus, Timmy is a carnivore (r) } conclusion
- This reasoning is (also) called *deduction*

The 1-Step Rules: Implication-Elimination and -Introduction

- **Elimination rule:**

$$\frac{A \rightarrow B \qquad A}{B} \rightarrow E$$

The 1-Step Rules: Implication-Elimination and -Introduction

- **Elimination rule:**

$$\frac{A \to B \qquad A}{B} \to E$$

- **Introduction rule:**

if we can derive B using A: then we can derive $A \to B$ and discharge A:

$$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B} \to I \quad \equiv \quad \begin{matrix}A \\ \vdots \\ B\end{matrix} \quad + \quad \frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B} \to I$$

The 1-Step Rules: Implication-Elimination and -Introduction (Based on Sequents)

- Derivation Rules as considered so far:

$$\frac{A \rightarrow B \qquad A}{B} \rightarrow E \qquad\qquad \frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \rightarrow B} \rightarrow I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

The 1-Step Rules: Implication-Elimination and -Introduction (Based on Sequents)

- Derivation Rules as considered so far:

$$\frac{A \rightarrow B \qquad A}{B} \rightarrow E \qquad\qquad \frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \rightarrow B} \rightarrow I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E \qquad\qquad \frac{X, A \vdash B}{\underbrace{X \vdash A \rightarrow B}_{\text{Has side effect of removing the assumption } A}} \rightarrow I$$

- We say that A gets *discharged*, and annotate that in the proof.

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \to B \quad \text{iff} \quad X, A \vdash B}_{\text{deduction equivalence (or deduction theorem)}}$$

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \to B$:
- If $X \vdash A \to B$, then $X, A \vdash B$:

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \to B \quad \text{iff} \quad X, A \vdash B}_{\substack{\text{deduction equivalence} \\ \text{(or deduction theorem)}}}$$

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \to B$: $$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$
- If $X \vdash A \to B$, then $X, A \vdash B$:

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \to B \quad \text{iff} \quad X, A \vdash B}_{\substack{\text{deduction equivalence} \\ \text{(or deduction theorem)}}}$$

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \to B$:
$$\frac{X, A \vdash B}{X \vdash A \to B}\to I$$

- If $X \vdash A \to B$, then $X, A \vdash B$:
$$\frac{X \vdash A \to B \qquad A \vdash A}{X, A \vdash B}\to E$$

  (That's the $\to E$ rule with Y substituted by A)

The 1-Step Rules: Implication-Introduction, Example 1

$p \vdash q \rightarrow (p \wedge q)$

α_1	(1)	p	A

The 1-Step Rules: Implication-Introduction, Example 1

$p \vdash q \rightarrow (p \wedge q)$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

α_1	(1)	p	A
α_1	(n)	$q \rightarrow (p \wedge q)$	

The 1-Step Rules: Implication-Introduction, Example 1

$p \vdash q \to (p \wedge q)$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(n-1)	$p \wedge q$	
α_1	(n)	$q \to (p \wedge q)$	(n-1)[α_2] $\to I$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption $\alpha_2 = q$.

The 1-Step Rules: Implication-Introduction, Example 1

$p \vdash q \rightarrow (p \wedge q)$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1, α_2	(n-1)	$p \wedge q$	
α_1	(n)	$q \rightarrow (p \wedge q)$	(n-1)[α_2] $\rightarrow I$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
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The 1-Step Rules: Implication-Introduction, Example 1

$p \vdash q \rightarrow (p \wedge q)$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1	(4)	$q \rightarrow (p \wedge q)$	3[α_2] $\rightarrow I$

α_1, α_2	(n-1)	$p \wedge q$	
α_1	(n)	$q \rightarrow (p \wedge q)$	(n-1)[α_2] $\rightarrow I$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption $\alpha_2 = q$.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$$p \to q \vdash (p \land r) \to q$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$$p \to q \vdash (p \wedge r) \to q$$

α_1	(1)	$p \to q$	A

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$p \to q \vdash (p \wedge r) \to q$

α_1	(1)	$p \to q$	A
α_1	(n)	$(p \wedge r) \to q$	

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$p \to q \vdash (p \wedge r) \to q$

α_1	(1)	$p \to q$	A
α_2	(2)	$p \wedge r$	A

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

α_1, α_2	(n-1)	q	
α_1	(n)	$(p \wedge r) \to q$	(n-1)[α_2] $\to I$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$p \to q \vdash (p \land r) \to q$

α_1	(1)	$p \to q$	A
α_2	(2)	$p \land r$	A
α_2	(3)	p	2 $\land E$
α_1, α_2	(n-1)	q	
α_1	(n)	$(p \land r) \to q$	(n-1)[α_2] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E$$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$$p \to q \vdash (p \land r) \to q$$

α_1	(1)	$p \to q$	A
α_2	(2)	$p \land r$	A
α_2	(3)	p	2 $\land E$
α_1, α_2	(4)	q	1,3 $\to E$
α_1, α_2	(n-1)	q	
α_1	(n)	$(p \land r) \to q$	(n-1)[α_2] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E$$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$$p \to q \vdash (p \land r) \to q$$

α_1	(1)	$p \to q$	A
α_2	(2)	$p \land r$	A
α_2	(3)	p	2 $\land E$
α_1, α_2	(4)	q	1,3 $\to E$
α_1	(5)	$(p \land r) \to q$	4[α_2] $\to I$

α_1, α_2	(n-1)	q	
α_1	(n)	$(p \land r) \to q$	(n-1)[α_2] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E$$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1 (cont'd)

The proof of $p \rightarrow q \vdash (p \wedge r) \rightarrow q$ in a tree-like structure:

$$\dfrac{\dfrac{p \rightarrow q \qquad \dfrac{[p \wedge r]^{(1)}}{p}\wedge E}{q}\rightarrow E}{(p \wedge r) \rightarrow q}\rightarrow I(1)$$

Here, we denote discharged assumptions by $[\ldots]^{(n)}$, where we number each assumption so that they can be distinguished from each other, i.e., so that we know which rule discharged which assumption(s).

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \land q) \to r \vdash p \to (q \to r)$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \land q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$$

α_1	(1)	$(p \land q) \rightarrow r$	A

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \wedge q) \to r \vdash p \to (q \to r)$$

α_1	(1)	$(p \wedge q) \to r$	A

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

α_1	(n)	$p \to (q \to r)$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

α_1	(1)	$(p \wedge q) \rightarrow r$	A
α_2	(2)	p	A

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

α_1, α_2	(n-1)	$q \rightarrow r$	
α_1	(n)	$p \rightarrow (q \rightarrow r)$	(n-1)[α_2] $\rightarrow I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \wedge q) \to r \vdash p \to (q \to r)$$

α_1	(1)	$(p \wedge q) \to r$	A
α_2	(2)	p	A
α_3	(3)	q	A
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \to r$	(n-2)[α_3] $\to I$
α_1	(n)	$p \to (q \to r)$	(n-1)[α_2] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \land q) \to r \vdash p \to (q \to r)$$

α_1	(1)	$(p \land q) \to r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \land q$	2,3 $\land I$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \to r$	(n-2)[α_3] $\to I$
α_1	(n)	$p \to (q \to r)$	(n-1)[α_2] $\to I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

α_1	(1)	$(p \wedge q) \rightarrow r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \wedge q$	2,3 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	1,4 $\rightarrow E$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \rightarrow r$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$p \rightarrow (q \rightarrow r)$	(n-1)[α_2] $\rightarrow I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

α_1	(1)	$(p \wedge q) \rightarrow r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \wedge q$	2,3 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	1,4 $\rightarrow E$
α_1, α_2	(6)	$q \rightarrow r$	5[α_3] $\rightarrow I$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \rightarrow r$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$p \rightarrow (q \rightarrow r)$	(n-1)[α_2] $\rightarrow I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$$(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$$

α_1	(1)	$(p \wedge q) \rightarrow r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \wedge q$	2,3 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	1,4 $\rightarrow E$
α_1, α_2	(6)	$q \rightarrow r$	5[α_3] $\rightarrow I$
α_1	(7)	$p \rightarrow (q \rightarrow r)$	6[α_2] $\rightarrow I$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \rightarrow r$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$p \rightarrow (q \rightarrow r)$	(n-1)[α_2] $\rightarrow I$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$p \vdash (q \to r) \to (q \to (p \wedge r))$

α_1	(1)	p	A

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$$p \vdash (q \to r) \to (q \to (p \wedge r))$$

α_1	(1)	p	A

α_1	(n)	$(q \to r) \to (q \to (p \wedge r))$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$p \vdash (q \to r) \to (q \to (p \wedge r))$

α_1	(1)	p	A
α_2	(2)	$q \to r$	A

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

α_1, α_2	(n-1)	$q \to (p \wedge r)$	
α_1	(n)	$(q \to r) \to (q \to (p \wedge r))$	(n-1)[α_2] $\to I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$$p \vdash (q \to r) \to (q \to (p \land r))$$

α_1	(1)	p	A
α_2	(2)	$q \to r$	A
α_3	(3)	q	A

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$p \land r$	
α_1, α_2	(n-1)	$q \to (p \land r)$	(n-2)[α_3] $\to I$
α_1	(n)	$(q \to r) \to (q \to (p \land r))$	(n-1)[α_2] $\to I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$$p \vdash (q \rightarrow r) \rightarrow (q \rightarrow (p \wedge r))$$

α_1	(1)	p	A
α_2	(2)	$q \rightarrow r$	A
α_3	(3)	q	A
α_2, α_3	(4)	r	2,3 $\rightarrow E$
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$p \wedge r$	
α_1, α_2	(n-1)	$q \rightarrow (p \wedge r)$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$(q \rightarrow r) \rightarrow (q \rightarrow (p \wedge r))$	(n-1)[α_2] $\rightarrow I$

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$$p \vdash (q \rightarrow r) \rightarrow (q \rightarrow (p \wedge r))$$

α_1	(1)	p	A
α_2	(2)	$q \rightarrow r$	A
α_3	(3)	q	A
α_2, α_3	(4)	r	2,3 $\rightarrow E$
$\alpha_1, \alpha_2, \alpha_3$	(5)	$p \wedge r$	1,4 $\wedge I$

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$p \wedge r$	
α_1, α_2	(n-1)	$q \rightarrow (p \wedge r)$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$(q \rightarrow r) \rightarrow (q \rightarrow (p \wedge r))$	(n-1)[α_2] $\rightarrow I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$$p \vdash (q \rightarrow r) \rightarrow (q \rightarrow (p \wedge r))$$

α_1	(1)	p	A
α_2	(2)	$q \rightarrow r$	A
α_3	(3)	q	A
α_2, α_3	(4)	r	2,3 $\rightarrow E$
$\alpha_1, \alpha_2, \alpha_3$	(5)	$p \wedge r$	1,4 $\wedge I$
α_1, α_2	(6)	$q \rightarrow (p \wedge r)$	5[α_3] $\rightarrow I$

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B} \rightarrow E$$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \rightarrow I$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$p \wedge r$	
α_1, α_2	(n-1)	$q \rightarrow (p \wedge r)$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$(q \rightarrow r) \rightarrow (q \rightarrow (p \wedge r))$	(n-1)[α_2] $\rightarrow I$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$$p \vdash (q \to r) \to (q \to (p \land r))$$

α_1	(1)	p	A
α_2	(2)	$q \to r$	A
α_3	(3)	q	A
α_2, α_3	(4)	r	2,3 $\to E$
$\alpha_1, \alpha_2, \alpha_3$	(5)	$p \land r$	1,4 $\land I$
α_1, α_2	(6)	$q \to (p \land r)$	5[α_3] $\to I$
α_1	(7)	$(q \to r) \to (q \to (p \land r))$	6[α_2] $\to I$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B} \to E$$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$p \land r$	
α_1, α_2	(n-1)	$q \to (p \land r)$	(n-2)[α_3] $\to I$
α_1	(n)	$(q \to r) \to (q \to (p \land r))$	(n-1)[α_2] $\to I$

Vacous Discharge: Discharging Non-existent Assumptions

- We can "discharge" assumptions that are not there; this happens if the conclusion does not depend on its assumption.

$$p \vdash q \to p$$

Vacous Discharge: Discharging Non-existent Assumptions

- We can "discharge" assumptions that are not there; this happens if the conclusion does not depend on its assumption.

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α_1, α_2	(n-1)	p	
α_1	(n)	$q \rightarrow p$	(n-1)[α_2] $\rightarrow I$

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α_1	(2)	$q \rightarrow p$	$1[]{\rightarrow}I$
$\alpha_1, \not\alpha_2$	(n-1)	p	
α_1	(n)	$q \rightarrow p$	(n-1)$[\not\alpha_2]\ {\rightarrow}I$

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- We call such a discharge a *vacuous discharge*.
- I.e., whenever we "would remove" some assumption α from a set of assumptions X, but $\alpha \notin X$, we write $i[]{\rightarrow}I$ instead of $i[\alpha]{\rightarrow}I$

Excursion: $\vdash$ vs. $\rightarrow$: An Often Asked Question in Previous Courses

- $\vdash$ and $\rightarrow$ seem to be of a very related nature:
  E.g., compare $A, B \vdash C$ with $A \wedge B \rightarrow C$
- So what's the difference?

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- Well, there are indeed very related, the difference is its technical meaning.
- $\vdash$ is used to introduce a *proof system* based on syntax manipulation. *It makes propositions about formulae.* It "states something" about interpretations (provided the sequent is valid).

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- $\vdash$ is used to introduce a *proof system* based on syntax manipulation. *It makes propositions about formulae.* It "states something" about interpretations (provided the sequent is valid).
- Indeed, you can write down any formula! Be it reasonable or not:
  - $q \rightarrow p$ and $\neg(q \rightarrow p)$.
  - $p \rightarrow (q \rightarrow p)$ and $\neg(p \rightarrow (q \rightarrow p))$
    (Note that the second holds some "truth" since it's a tautology)
- So in conclusion, sequents *relate* formulae. A formula itself doesn't mean anything, it's just a formula.

Theorems

Unconditionally True Formulas

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- Thus, we get $\vdash p \rightarrow (q \rightarrow (p \wedge q))$, so its formula is a theorem.
- Note that A in $\vdash A$ is a tautology!

From Arbitrary Sequents to Theorems

- Recall our example from the last couple of slides:
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  - (we claimed that) $\vdash p \rightarrow (q \rightarrow p)$
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- This means we can just “move” all assumptions as antecedents into the formula! (Just apply that equivalence recursively.)

Summary

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→ The Logic Notes sections:
  - *3. More about propositional logic*: *Truth Tables*
  - *2. Propositional natural deduction*: *Conjunction*
  - *2. Propositional natural deduction*: *Implication*
  - *2. Propositional natural deduction*: *Counting assumptions* (except *Contraction*, which you should look up!)