Logo (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Sequents, Semantics, and Propositional Natural Deduction — Conjunction, Implication, Theorems

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Introduction

Recap: Connectives and Formulae: Syntax

The main connective dictates the type of a formula:

- if main connective is $\neg$, formula is a *negation*
- ... $\wedge$, ... *conjunction*
- ... $\vee$, ... *disjunction*
- ... $\rightarrow$, ... *implication*
- ... $\leftrightarrow$, ... *double-implication*

Recap: Connectives and Formulae: Semantics

What do these connectives *mean*?

- The "intended meaning" of connectives is expressed by truth tables:

p	$\neg$
0	1
1	0

p	q	$\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$\vee$
0	0	0
0	1	1
1	0	1
1	1	1

p	q	$\rightarrow$
0	0	1
0	1	1
1	0	0
1	1	1

- The truth value of a formula ϕ is defined by evaluating the formula under a given *interpretation*, which is an assignment of all propositional symbols.

Sequents and Natural Deduction: What and Why?: Sequents

- We want to know when one logical formula follows logically from another.
- Suppose we know that "p is true', e.g., due to some observation (technically: thus know that it is *interpreted* as true), and we know that $p \to q$ holds as well. Then we can logically conclude that q also holds!
- We can express this with *sequents*: $p, p \to q \models q$
- These conclusions can be arbitrarily complicated, however! I.e., it might not be obvious that the conclusion follows from the premises.
- We use Natural Deduction to "manipulate sequents" step-wise thus "showing" validity.

Sequents

Introduction

Our convention:

- Letters from the *end* of the alphabet: set
- ... *beginning* ... : single object of the kind that's in the set

This represents a *valid* sequent: $X \models A$

- Read it: Formula A follows (logically) from the formulae in X
- For example, "q follows from p and $p \to q$"
- We write down: $p, p \to q \models q$
  but that's just short for: $\underbrace{\{p, p \to q\}}_{X} \models \underbrace{q}_{A}$
- Also $X, Y \models A$ is short for $X \cup Y \models A$,
  $X, B \models A$ is short for $X \cup \{B\} \models A$, and
  $X, B, C \models A$ is short for $X \cup \{B, C\} \models A$.

Another Example for a Valid Sequent

Previous example: $p, p \to q \models q$

But what if the conclusion isn't a "true" proposition (i.e., that's interpreted by 1)? What if it's a formula? What would that mean?

What does it mean for a *formula* to follow logically?

- Assume we know $a \wedge (b \vee c)$ "holds", does $(b \vee c)$ follow as well?
- What does this even *mean*? We don't have the *property* "hold"?!

Answers:

- Yes, $a \wedge (b \vee c) \models (b \vee c)$ holds, i.e., it's a valid sequent!
- The formal definition is based on interpretations.

Semantically Valid Sequents

Definition:
$X \models A$ means the sequent is *valid*. This is the case if and only if:

- *A is true for every interpretation for which all the formulae in X are true.* Or, equivalently:
- *There is no interpretation that makes X true, but not A.*

How to check/test/prove $X \models A$? Create the proof tables!

- Create a table t_X for all formulae in X (all need to be true)
- Create another table t_A for A and check validity criterion.

Checking Validity, Example 1

How to prove $a \wedge (b \vee c) \models (b \vee c)$? *done live*

This is done only in an additional (offline) recording.

But you should be able to do this yourself, also based on the second example on the next slide!

If you still have trouble, watch the lecture recording.

Checking Validity, Example 2

Show $\overbrace{(p \vee q) \rightarrow r, p}^{X} \models \overbrace{(p \rightarrow r) \wedge (q \rightarrow r)}^{A}$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \rightarrow r$	$q \rightarrow r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Recall the definition: The sequent is valid if all interpretations that make X true also make A true!

Checking Validity, Example 2

Show $\overbrace{(p \vee q) \rightarrow r, p}^{X} \models \overbrace{(p \rightarrow r) \wedge (q \rightarrow r)}^{A}$

Table t_X for premises:

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	X
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Table t_A for conclusion:

p	q	r	$p \rightarrow r$	$q \rightarrow r$	A
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Only two interpretations exist that make all $x \in X$ true:

1. $I_1(p) = I_1(r) = 1, I_1(q) = 0$
2. $I_2(p) = I_2(q) = I_2(r) = 1$

Both of them make A true! Thus, $X \models A$.

Natural Deduction

Motivation

So, it's all about finding out finding out whether some formula follows logically from the interaction of many others!

- E.g. you might have a *huge* knowledge base *KB* of rules. Maybe a medical database with (certified, based on experience or research) rules stating which symptoms indicate diseases or affected organs etc.
- What if we have a hypothesis about another rule that's not yet in the system?
  E.g., *if symptom p is present, it cannot be disease q.* If that's true, it would mean $KB \models p \rightarrow \neg q$.

So we can just use truth tables and we are done, right?

Well, in theory, yes. But ... efficiency!

Inefficiency of Proving Validity with Truth Tables

- Note that truth tables *always* grow exponentially, not just in the worst case!
- Assume we want to show $X \models A$ and the number of propositional symbols is:
  - $1 \Rightarrow 2$ lines (like the $\neg$ truth table)
  - $2 \Rightarrow 4$ lines (like $\wedge$, $\vee$, and $\rightarrow$)
  - $3 \Rightarrow 8$ lines (like our previous example!)
  - $4 \Rightarrow 16$ lines
  - $10 \Rightarrow 1.024$ lines
  - $20 \Rightarrow 1.048.576$ lines ($>$ 1 million)
  - $n \Rightarrow 2^n$ lines (i.e., number of interpretations)

$\Rightarrow$ We need a proof system that is not (always) that bad!
  - We start with Natural Deduction!

Natural Deduction and Derivations

- Natural deduction is pure syntax manipulation and acts as *proof system*.
- Natural Deduction exploits *derivations*.
- A derivation is a finite sequence of formulae, which are derived from each other based on elementary formula manipulations ("1-step inference rules")
- For each connective we will use two rules: one for *introducing* it, and one for *eliminating* it.

Syntax of Sequents

- From now on, we write $X \vdash A$ rather than $X \models A$.
- The reason is that $X \models A$ denotes that A follows logically from X, but usually we still want to find that out using some proof system.
- E.g., we could also write down $p \vdash \neg p$, which is an *invalid* sequent.
- So, if some sequent $X \vdash A$ is given, we are interested in finding out whether it is actually *valid*, denoted by $X \models A$.
- To show that it's valid, we use Natural deduction.
- Only in the second part you will formally learn the relationship between these two concepts $\vdash$ and $\models$. I.e., you will learn how to show that each sequent $X \vdash A$ that is proved by some proof system (like Natural Deduction) is actually valid.

Conjunction

The 1-Step Rules: And-Elimination

What are the 1-step rules for dealing with conjunction?

Elimination rule:

$$\frac{A \wedge B}{A}\wedge E \qquad\qquad \frac{A \wedge B}{B}\wedge E$$

Which reads: If we derived $A \wedge B$, we can derive both A and B.

The 1-Step Rules: And-Introduction

What are the 1-step rules for dealing with conjunction?

Introduction rule:

$$\frac{A \qquad B}{A \wedge B}\wedge I$$

Which reads: If we derived A and we derived B, we can derive $A \wedge B$.

Proof Syntax / Notation: Overview

- How to write down proofs?
- There are many different notations that describe the same thing
- We introduce two:
  - Tree-like representation of the applied rules (just since it's another standard)
  - list-like representation (only use that one!)

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$
- Sequence of derivations: $\underbrace{p \wedge q}_{\text{premise}}, \underbrace{q}_{\wedge E}, \underbrace{p}_{\wedge E}, \underbrace{q \wedge p}_{\wedge I \text{ and conclusion}}$
- In the tree-like format:

$$\dfrac{\dfrac{p \wedge q}{q}\wedge E \qquad \dfrac{p \wedge q}{p}\wedge E}{q \wedge p}\wedge I$$

- Leaves are assumptions, root is conclusion
- Advantages: Makes the proof structure obvious
- In exercises, etc: *Do not use it, unless we ask you to!*

Proof Syntax / Notation: Tree- and List-like Representations

- Assume we want to prove $p \wedge q \vdash q \wedge p$
- Sequence of derivations: $\underbrace{p \wedge q}_{\text{premise}}, \underbrace{q}_{\wedge E}, \underbrace{p}_{\wedge E}, \underbrace{q \wedge p}_{\wedge I \text{ and conclusion}}$
- In the list format:

α_1	(1)	$p \wedge q$	A	$\equiv$	$p \wedge q \vdash p \wedge q$
α_1	(2)	q	1 $\wedge E$	$\equiv$	$p \wedge q \vdash q$
α_1	(3)	p	1 $\wedge E$	$\equiv$	$p \wedge q \vdash p$
α_1	(4)	$q \wedge p$	2,3 $\wedge I$	$\equiv$	$p \wedge q \vdash q \wedge p$

column 1: assumption number *column 2:* line number
column 3: derivation *column 4:* how it was derived

- **Note:** Each line represents a sequent! (Sequence of sequents.)

The 1-Step Rules (Based on Sequents): Derivation Rules

- Derivation Rules as considered so far:

$$\dfrac{A \wedge B}{A}\wedge E \qquad \dfrac{A \wedge B}{B}\wedge E \qquad \dfrac{A \quad B}{A \wedge B}\wedge I$$

- Re-written in terms of sequents:

$$\dfrac{X \vdash A \wedge B}{X \vdash A}\wedge E \qquad \dfrac{X \vdash A \wedge B}{X \vdash B}\wedge E \qquad \dfrac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B}\wedge I$$

$\rightarrow$ I.e., now we see how premises accumulate!

The 1-Step Rules (Based on Sequents): Accumulation of Assumptions, Example

$p, q \vdash p \wedge q$

α_1	(1)	p	A	$\equiv$	$p \vdash p$ (by assumption)
α_2	(2)	q	A	$\equiv$	$q \vdash q$ (by assumption)
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$	$\equiv$	$p, q \vdash p \wedge q$ $(\wedge I)$

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B}\wedge I$$

Implication

Introduction

- Now we consider the "if . . . , then . . . " connective: implication!
- E.g.,
  - $p \rightarrow q$: "if it is raining (p), then the ground is wet (q)"
  - Here, p is the *antecedent* and q the *consequent*
  - $(p \wedge q) \rightarrow r$:
    - All tigers are carnivores (p) } premises
    - Timmy is a tiger (q) } premises
    - Thus, Timmy is a carnivore (r) } conclusion
- This reasoning is (also) called *deduction*

The 1-Step Rules: Implication-Elimination and -Introduction

- **Elimination rule:**

$$\frac{A \rightarrow B \qquad A}{B}\rightarrow E$$

- **Introduction rule:**

if we can derive B using A: then we can derive $A \rightarrow B$ and discharge A:

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow B}\rightarrow I \quad \equiv \quad \begin{matrix}A\\ \vdots \\ B\end{matrix} \quad + \quad \frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow B}\rightarrow I$$

The 1-Step Rules: Implication-Elimination and -Introduction (Based on Sequents)

- Derivation Rules as considered so far:

$$\frac{A \rightarrow B \qquad A}{B}\rightarrow E \qquad\qquad \frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow B}\rightarrow I$$

- Re-written in terms of sequents:

$$\frac{X \vdash A \rightarrow B \qquad Y \vdash A}{X, Y \vdash B}\rightarrow E \qquad\qquad \underbrace{\frac{X, A \vdash B}{X \vdash A \rightarrow B}\rightarrow I}_{\text{Has side effect of removing the assumption } A}$$

- We say that A gets *discharged*, and annotate that in the proof.

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \rightarrow B \quad \text{iff} \quad X, A \vdash B}_{\text{deduction equivalence (or deduction theorem)}}$$

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \rightarrow B$: $\quad \frac{X, A \vdash B}{X \vdash A \rightarrow B}\rightarrow I$
- If $X \vdash A \rightarrow B$, then $X, A \vdash B$: $\quad \frac{X \vdash A \rightarrow B \qquad A \vdash A}{X, A \vdash B}\rightarrow E$

  (That's the $\rightarrow E$ rule with Y substituted by A)

Introduction Sequents Natural Deduction Conjunction Implication Theorems Summary

The 1-Step Rules: Deduction Equivalence

$$\underbrace{X \vdash A \rightarrow B \quad \text{iff} \quad X, A \vdash B}_{\text{deduction equivalence (or deduction theorem)}}$$

Why does this hold?

- If $X, A \vdash B$, then $X \vdash A \rightarrow B$: $\quad \frac{X, A \vdash B}{X \vdash A \rightarrow B}\rightarrow I$
- If $X \vdash A \rightarrow B$, then $X, A \vdash B$: $\quad \frac{X \vdash A \rightarrow B \qquad A \vdash A}{X, A \vdash B}\rightarrow E$

  (That's the $\rightarrow E$ rule with Y substituted by A)

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The 1-Step Rules: Implication-Introduction, Example 1

$p \vdash q \rightarrow (p \wedge q)$

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B}\rightarrow I$$

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(3)	$p \wedge q$	1,2 $\wedge I$
α_1	(4)	$q \rightarrow (p \wedge q)$	3[α_2] $\rightarrow I$

α_1, α_2	(n-1)	$p \wedge q$	
α_1	(n)	$q \rightarrow (p \wedge q)$	(n-1)[α_2] $\rightarrow I$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption $\alpha_2 = q$.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1

$p \to q \vdash (p \land r) \to q$

α_1	(1)	$p \to q$	A
α_2	(2)	$p \land r$	A
α_2	(3)	p	2 $\land E$
α_1, α_2	(4)	q	1,3 $\to E$
α_1	(5)	$(p \land r) \to q$	4[α_2] $\to I$
α_1, α_2	(n-1)	q	
α_1	(n)	$(p \land r) \to q$	(n-1)[α_2] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B}\to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B}\to E$$

- Assumption α_2 is a new one, which was not given in the original sequent, so we need to eliminate it later on.
- In the last step, we discharge assumption α_2.

The 1-Step Rules: Implication-Introduction and -Elimination, Example 1 (cont'd)

The proof of $p \to q \vdash (p \land r) \to q$ in a tree-like structure:

$$\dfrac{\dfrac{p \to q \qquad \dfrac{[p \land r]^{(1)}}{p}\land E}{q}\to E}{(p \land r) \to q}\to I(1)$$

Here, we denote discharged assumptions by $[\ldots]^{(n)}$, where we number each assumption so that they can be distinguished from each other, i.e., so that we know which rule discharged which assumption(s).

The 1-Step Rules: Implication-Introduction and -Elimination, Example 2

$(p \land q) \to r \vdash p \to (q \to r)$

α_1	(1)	$(p \land q) \to r$	A
α_2	(2)	p	A
α_3	(3)	q	A
α_2, α_3	(4)	$p \land q$	2,3 $\land I$
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	1,4 $\to E$
α_1, α_2	(6)	$q \to r$	5[α_3] $\to I$
α_1	(7)	$p \to (q \to r)$	6[α_2] $\to I$
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	r	
α_1, α_2	(n-1)	$q \to r$	(n-2)[α_3] $\to I$
α_1	(n)	$p \to (q \to r)$	(n-1)[α_2] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B}\to I$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B}\to E$$

The 1-Step Rules: Implication-Introduction and -Elimination, Example 3

$p \vdash (q \to r) \to (q \to (p \land r))$

α_1	(1)	p	A
α_2	(2)	$q \to r$	A
α_3	(3)	q	A
α_2, α_3	(4)	r	2,3 $\to E$
$\alpha_1, \alpha_2, \alpha_3$	(5)	$p \land r$	1,4 $\land I$
α_1, α_2	(6)	$q \to (p \land r)$	5[α_3] $\to I$
α_1	(7)	$(q \to r) \to (q \to (p \land r))$	6[α_2] $\to I$
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$p \land r$	
α_1, α_2	(n-1)	$q \to (p \land r)$	(n-2)[α_3] $\to I$
α_1	(n)	$(q \to r) \to (q \to (p \land r))$	(n-1)[α_2] $\to I$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X, Y \vdash B}\to E$$

$$\frac{X, A \vdash B}{X \vdash A \to B}\to I$$

Vacous Discharge: Discharging Non-existent Assumptions

- We can "discharge" assumptions that are not there; this happens if the conclusion does not depend on its assumption.

$p \vdash q \to p$

~~α_1~~ α_1	(1)	p	A
~~α_2~~	~~(2)~~	~~q~~	~~A~~
α_1	(2)	$q \to p$	1[]$\to I$

α_1, ~~α_2~~	(n-1)	p	
α_1	(n)	$q \to p$	(n-1)[~~α_2~~] $\to I$

$$\frac{X, A \vdash B}{X \vdash A \to B} \to I$$

- We call such a discharge a *vacuous discharge*.
- I.e., whenever we "would remove" some assumption α from a set of assumptions X, but $\alpha \notin X$, we write $i[]\to I$ instead of $i[\alpha]\to I$

Excursion: $\vdash$ vs. $\to$: An Often Asked Question in Previous Courses

- $\vdash$ and $\to$ seem to be of a very related nature: E.g., compare $A, B \vdash C$ with $A \land B \to C$
- So what's the difference?
- Well, there are indeed very related, the difference is its technical meaning.
- $\vdash$ is used to introduce a *proof system* based on syntax manipulation. *It makes propositions about formulae.* It "states something" about interpretations (provided the sequent is valid).
- Indeed, you can write down any formula! Be it reasonable or not:
  - $q \to p$ and $\neg(q \to p)$.
  - $p \to (q \to p)$ and $\neg(p \to (q \to p))$ (Note that the second holds some "truth" since it's a tautology)
- So in conclusion, sequents *relate* formulae. A formula itself doesn't mean anything, it's just a formula.

Theorems

Unconditionally True Formulas

- Sequents that do not depend on anything are called *theorems*!
- Thus, A is a theorem if "$\vdash A$", e.g., $\vdash p \to (q \to p)$.
- Another (slightly more complex) example:

α_1	(1)	p	A
α_2	(2)	q	A
α_1, α_2	(3)	$p \land q$	1,2 $\land I$
α_1	(4)	$q \to (p \land q)$	3[α_2] $\to I$
	(5)	$p \to (q \to (p \land q))$	4[α_1] $\to I$

- Thus, we get $\vdash p \to (q \to (p \land q))$, so its formula is a theorem.
- Note that A in $\vdash A$ is a tautology!

From Arbitrary Sequents to Theorems

- Recall our example from the last couple of slides:
  - We proved $p \vdash q \to p$, and
  - (we claimed that) $\vdash p \to (q \to p)$
- We can generalize this to obtain arbitrarily many theorems! How?
- Remember the *deduction equivalence*!

$$X \vdash A \to B \quad \text{iff} \quad X, A \vdash B$$

- This means we can just "move" all assumptions as antecedents into the formula! (Just apply that equivalence recursively.)

Summary

Content of this Lecture

- Sequents and their semantics
  - What does $X \models A$ mean?
- The most important basics of Natural Deduction!
  - How can proofs be written?
  - What does $X \vdash A$ mean?
  - Every logical connective comes with two 1-step rules: *Introduction* and *Elimination*
  - What's a theorem?

→ The Logic Notes sections:
  - *3. More about propositional logic*: *Truth Tables*
  - *2. Propositional natural deduction*: *Conjunction*
  - *2. Propositional natural deduction*: *Implication*
  - *2. Propositional natural deduction*: *Counting assumptions* (except *Contraction*, which you should look up!)