Logic (PHIL 2080, COMP 2620, COMP 6262) Chapter: Propositional Natural Deduction — Negation, Disjunction

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Introduction



• What are theorems? (Sequents without assumptions!)



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- Relationship between \vdash and \rightarrow :
 - They live in completely different worlds!
 - \rightarrow is a connective and thus part of a formula, just like \neg , \land , and \lor .
 - ⊢ is *not* a connective and can thus not possibly be part of *any* formula! It only states whether we can derive a single formula *A* from a set of formulae *X*, expressed by *X* ⊢ *A*.



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 - Conjunction (easy!)
 - Implication (not quite that easy!)
- So what's missing?
 - Negation (not as easy as you might think!)
 - Disjunction (quite hard... Practice it!)



Negation



	Negation ○● ○ ○○○○○○○○○○	
Introduction:	Intuitive Meaning	

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- Be careful when translating "not" used in natural language:
 - Someone likes Logic (= p)
 - Someone doesn't like Logic! ($\neq \neg p$)
 - Such complex propositions will be covered in predicate logic!









$$\begin{array}{c|c}
p & \neg p \\
\bullet & 0 & 1 \\
1 & 0
\end{array}$$









• I.e., in propositional logic, two negations eliminate each other!





- I.e., in propositional logic, two negations eliminate each other!
- It's not true that it's not true that Socrates is a goat (So it is true!)





The 1-Step Rules: Double-Negation Elimination and Introduction

- The (second) truth table gives us the following two rules:
- Double-Negation Elimination and Introduction Rules:

$$\frac{\neg \neg A}{A} \neg \neg E \qquad \qquad \frac{A}{\neg \neg A} \neg \neg I$$



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- Double-Negation Elimination and Introduction Rules:

$$\frac{\neg \neg A}{A} \neg \neg E \qquad \qquad \frac{A}{\neg \neg A} \neg \neg b$$

Again based on sequents:

$$\frac{X \vdash \neg \neg A}{X \vdash A} \neg \neg E \qquad \qquad \frac{X \vdash A}{X \vdash \neg \neg A} \neg \neg I$$



Avoid the next common mistake:

- Look carefully what/where the main connective is!
- The rule refers to a *complete formula*!
- So, e.g., we cannot go from $p \land \neg \neg q$ to $p \land q$ in just one step!

$$\frac{\neg \neg A}{A} \neg \neg E \qquad \frac{A}{\neg \neg A} \neg \neg I$$

Because $A = (p \land \neg \neg q)$, but the rule states it should be $\neg \neg q!$



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 α_1 (3) $\neg \neg p$ 2 $\neg \neg l$



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The 1-Step Rules: Negation-Elimination

- With the double-negation rules we can't introduce or eliminate a *single* negation.
- To deal with single negations, we require the symbol \perp .
- We introduced it before: it represents "false", an "absurd" constant that can never be satisfied.



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- To deal with single negations, we require the symbol \perp .
- We introduced it before: it represents "false", an "absurd" constant that can never be satisfied.
- Negation-Elimination rule: (without and with sequent-notation)

$$\frac{A \quad \neg A}{\bot} \neg E \qquad \qquad \frac{X \vdash A \quad Y \vdash \neg A}{X, Y \vdash \bot} \neg E$$



Negation ○○○○○●○○○○○	

The 1-Step Rules: Negation-Introduction

Negation-Introduction rule:

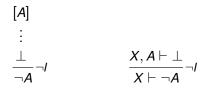






The 1-Step Rules: Negation-Introduction

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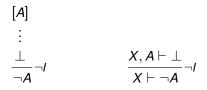


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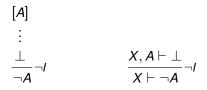


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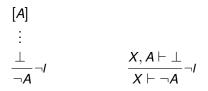


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 - This rule is the main proof idea behind the proof technique "Proof by contradiction". (There are, e.g., nice illustrations on YouTube proving that $\sqrt{2}$ is not rational by that technique.)



• We want to show: If you are in Canberra (*p*), you are not in Sydney($\neg q$); thus: if you are in Sydney (*q*), you are not in Canberra ($\neg p$) I.e., $p \rightarrow \neg q \vdash q \rightarrow \neg p$



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 - Assume the premise (i.e., p
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 - Thus, "You are in Sydney and in Canberra", I(q) = 1 and I(p) = 1

Why?

р	q	$\neg p$	q ightarrow eg p	eg(q ightarrow eg p)
0	0 1 0 1	1	1	0
0	1	1	1	0
1	0	0	1	0
1	1	0	0	1



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 You are in Sydney(q)
 If you are in Canberra (p), you are not in Sydney(¬q); thus:
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 - Because of the first assumption, and since we just assumed we are in Canberra, we can conclude that we are not in Sydney.



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 - Because of the first assumption, and since we just assumed we are in Canberra, we can conclude that we are not in Sydney.
 - But now we are in Sydney, and not in Sydney, contradiction!
 - Thus our additional assumption that the second implication is false must be wrong, so it must be true!
 - Thus, the first implication implies the second! q.e.d.





If you are in Canberra (*p*), you are not in Sydney (¬*q*); thus:
 if you are in Sydney (*q*), you are not in Canberra (¬*p*)

 $p
ightarrow \neg q dash q
ightarrow \neg p$



The 1-Step Rules: Negation-Elimination and -Introduction, Example 1

$$p \to \neg q \vdash q \to \neg p$$

$$\alpha_1 \qquad (1) \quad p \to \neg q \quad \mathsf{A}$$



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$$\alpha_1$$
 (n) $q \rightarrow \neg p$



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$$\alpha_1 \qquad (1) \quad p \rightarrow \neg q \quad A$$

$$\alpha_2 \qquad (2) \quad q \qquad A$$



$$\begin{array}{lll} \alpha_1, \alpha_2 & (n\text{-}1) & \neg p \\ \alpha_1 & (n) & q \to \neg p & (n\text{-}1) \to I \end{array}$$



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$$\alpha_1 \qquad (1) \quad p \rightarrow \neg q \quad A$$

$$\alpha_2 \qquad (2) \quad q \quad A$$

$$\alpha_3 \qquad (3) \quad p \quad A$$

$$\boxed{\frac{X,A\vdash\bot}{X\vdash\neg A}\neg l}$$

$$\begin{array}{cccc} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-2}) & \bot \\ \alpha_1, \alpha_2 & (\mathsf{n-1}) & \neg p & (\mathsf{n-2})[\alpha_3] \neg I \\ \alpha_1 & (\mathsf{n}) & q \to \neg p & (\mathsf{n-1}) \to I \end{array}$$



Negation

The 1-Step Rules: Negation-Elimination and -Introduction, Example 1

• If you are in Canberra (p), you are not in Sydney $(\neg q)$; thus: if you are in Sydney (q), you are not in Canberra $(\neg p)$

$$p
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ightarrow \neg p$$

 $lpha_1$ (1) $p
ightarrow \neg q$ A
 $lpha_2$ (2) q A
 $lpha_3$ (3) p A

$$\frac{X\vdash A \quad Y\vdash \neg A}{X, Y\vdash \bot}\neg E$$

$$\frac{X,A\vdash\bot}{X\vdash\neg A}\neg I$$

$$\begin{array}{cccc} \alpha_1, \alpha_2, \alpha_3 & (\textbf{n-2}) & \perp & \textbf{x}, \textbf{y} \neg \textbf{E} \\ \alpha_1, \alpha_2 & (\textbf{n-1}) & \neg p & (\textbf{n-2})[\alpha_3] \neg \textbf{I} \\ \alpha_1 & (\textbf{n}) & q \rightarrow \neg p & (\textbf{n-1}) \rightarrow \textbf{I} \end{array}$$

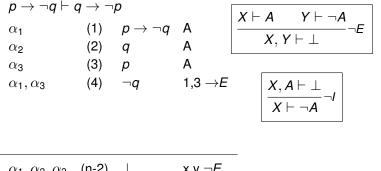
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troduction Negation Disjunctions Summary

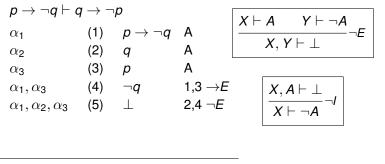
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$$\begin{array}{cccc} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-2}) & \bot & \mathsf{x,y} \neg \mathsf{E} \\ \alpha_1, \alpha_2 & (\mathsf{n-1}) & \neg \mathsf{p} & (\mathsf{n-2})[\alpha_3] \neg \mathsf{I} \\ \alpha_1 & (\mathsf{n}) & q \rightarrow \neg \mathsf{p} & (\mathsf{n-1}) \rightarrow \mathsf{I} \end{array}$$



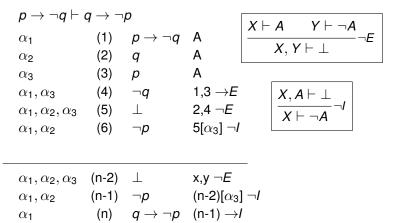
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$$\begin{array}{lll} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-2}) & \bot & \mathsf{x}, \mathsf{y} \neg E \\ \alpha_1, \alpha_2 & (\mathsf{n-1}) & \neg p & (\mathsf{n-2})[\alpha_3] \neg I \\ \alpha_1 & (\mathsf{n}) & q \rightarrow \neg p & (\mathsf{n-1}) \rightarrow I \end{array}$$



The 1-Step Rules: Negation-Elimination and -Introduction, Example 1





The 1-Step Rules: Negation-Elimination and -Introduction, Example 1

p ightarrow eg q dash	$q ightarrow \neg$	р	X	$\vdash A Y \vdash \neg A$
α_1	(1)	$p ightarrow \neg q$	A $\begin{vmatrix} x \\ - \end{vmatrix}$	¬E
α_2	(2)	q	A	$X, Y \vdash \bot$
α_3	(3)	р	А	
α_1, α_3	(4)	eg q	1,3 <i>→E</i>	$X, A \vdash \bot$
$\alpha_1, \alpha_2, \alpha_3$	(5)	\perp	2,4 <i>⊢E</i>	$\overline{X \vdash \neg A} \neg I$
α_1, α_2	(6)	$\neg p$	5[α₃] ¬/	
α_1	(7)	$q ightarrow \neg p$	$6[\alpha_2] \rightarrow I$	
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	\perp	x,y <i>¬E</i>	
α_1, α_2	(n-1)	eg p	(n-2)[<i>α</i> ₃] ¬ <i>l</i>	1
α_1	(n)	q ightarrow eg p	(n-1) →/	





- Contradict yourself, and I don't care anymore!
- In other words: We can conclude all we want from an inconsistent knowledge base.

 $p, \neg p \vdash q$



Negation	
000000000000	
	-

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 α_1 (1) p A



Negation ○○○○○○○○●○○○	

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$$\begin{array}{cccc} \alpha_1 & (1) & p & \mathsf{A} \\ \alpha_2 & (2) & \neg p & \mathsf{A} \end{array}$$

$$\frac{X\vdash A \quad Y\vdash \neg A}{X, Y\vdash \bot}\neg E$$



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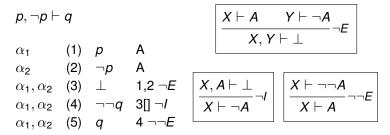
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 Here we have another example of vacuous discharge: We blame the contradiction on a non-existing assumption ¬q.



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- A 2-Step Rule: Reductio ad Absurdum (RAA)
 - We can combine Negation-Elimination with its Introduction: Again, notations without and with sequents:

$$\begin{bmatrix} B \end{bmatrix} \qquad \begin{bmatrix} B \end{bmatrix}$$

$$\vdots \qquad \vdots$$

$$\frac{A \qquad \neg A}{\neg B} RAA \qquad \frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

• The rules discharge assumption *B*.



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- The rules discharge assumption B.
- Why is it correct? $\frac{X, A \vdash \bot}{X \vdash \neg A} \neg I \quad \frac{X \vdash A}{X, Y \vdash \bot} \gamma \vdash \neg A}{X, Y \vdash \bot} \neg E \quad \frac{X, B \vdash A \quad Y, B \vdash \neg A}{X, Y, B \vdash \bot} \neg E}{X, Y \vdash \neg B} \neg I$



• $p \rightarrow \neg p \vdash \neg p$: *p* is *so false*, it implies its own negation!



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 $p \rightarrow \neg p \vdash \neg p$



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$$p \rightarrow \neg p \vdash \neg p$$

$$\alpha_1$$
 (1) $p \rightarrow \neg p$ A



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$$p \rightarrow \neg p \vdash \neg p$$

 α_1 (1) $p \rightarrow \neg p$ A

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

$$\alpha_1$$
 (n) $\neg p$



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$$p \rightarrow \neg p \vdash \neg p$$

$$\alpha_1 \qquad (1) \quad p \rightarrow \neg p \quad \mathsf{A}$$

$$\alpha_2 \qquad (2) \quad p \qquad \mathsf{A}$$

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

$$\alpha_1$$
 (n) $\neg p$ x,y[α_2] RAA



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$$\alpha_{1} \quad (1) \quad p \rightarrow \neg p \quad A$$

$$\alpha_{2} \quad (2) \quad p \quad A$$

$$\alpha_{1}, \alpha_{2} \quad (3) \quad \neg p \quad 1, 2 \rightarrow E$$

$$\alpha_{1} \quad (n) \quad \neg p \quad x, y[\alpha_{2}] \quad RAA$$



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ho ightarrow eg ho	ho ightarrow eg ho arrow eg ho arrow eg ho				<i>Y</i> , <i>B</i> ⊢	$\neg A$
α_1	(1)	p ightarrow eg p	А	X,	$Y \vdash \neg B$	— паа
α_2	(2)	р	А			
α_1, α_2	(3)	$\neg p$	1,2 $ ightarrow$	E		
α_1	(4)	$\neg p$	2,3[$lpha_2$] RAA		
α_1	(n)	$\neg p$	x,y[$lpha_2$] RAA		





• $\neg p \rightarrow p \vdash p$: if *p* is even implied by its own negation, then it *must* be true!



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$$eg p
ightarrow p
ightarrow p$$

 $lpha_1$ (1) $eg p
ightarrow p$ A



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$$eg p \to p \vdash p$$

$$\alpha_1$$
 (1) $\neg p \rightarrow p$ A

$$\alpha_2$$
 (2) $\neg p$ A



• $\neg p \rightarrow p \vdash p$: if *p* is even implied by its own negation, then it *must* be true!

$$\neg p \rightarrow p \vdash p \qquad \qquad \qquad \frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

$$\alpha_1 \qquad (1) \quad \neg p \rightarrow p \quad A$$

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$$\alpha_1, \alpha_2 \qquad (3) \quad p \qquad \qquad 1, 2 \rightarrow E$$



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$$\alpha_1, \alpha_2 \qquad (3) \quad p \qquad 1, 2 \rightarrow E$$

$$\alpha_1 \qquad (4) \quad \neg \neg p \qquad 2, 3[\alpha_2] RAA$$



• $\neg p \rightarrow p \vdash p$: if *p* is even implied by its own negation, then it *must* be true!

$$\neg p \rightarrow p \vdash p$$

$$\begin{array}{c} X, B \vdash A \quad Y, B \vdash \neg A \\ \hline X, Y \vdash \neg B \end{array} RAA$$

$$\begin{array}{c} \alpha_1 & (1) \quad \neg p \rightarrow p \quad A \\ \alpha_2 & (2) \quad \neg p \quad A \\ \alpha_1, \alpha_2 & (3) \quad p \quad 1, 2 \rightarrow E \\ \alpha_1 & (4) \quad \neg \neg p \quad 2, 3[\alpha_2] RAA \\ \alpha_1 & (5) \quad p \quad 4 \neg \neg E \end{array}$$



Disjunctions



	Disjunctions ○●○○○○○○○	

Introduction: (Our) Or versus Exclusive Or

- Disjunctions are of the form $A \vee B$
- It rains this afternoon or this evening. (But it can also be both!)



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 (Unless it's a physicist's cat, the choice is *exclusive*! The cat *cannot be both dead and alive*!)



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Introduction: (Our) Or versus Exclusive Or

- Disjunctions are of the form $A \vee B$
- It rains this afternoon or this evening. (But it can also be both!)
- The cat is *either* dead *or* alive.
 (Unless it's a physicist's cat, the choice is *exclusive*! The cat *cannot be both dead and alive*!)
- We use the first, non-exclusive, notion of or: At least one proposition needs to be true!



	Disjunctions ○○●○○○○○○○	

Disjunction-Introduction Rules:

• Notation without sequents:

$$\frac{A}{A \lor B} \lor I \qquad \qquad \frac{B}{A \lor B} \lor I$$



	Disjunctions ○○●○○○○○○○	

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• Notation with sequents:

$$\frac{X \vdash A}{X \vdash A \lor B} \lor I \qquad \qquad \frac{X \vdash B}{X \vdash A \lor B} \lor I$$



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Great! So we have that easy rule to prove a disjunction, right?



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- Great! So we have that easy rule to prove a disjunction, right?
- Well... No. (That's only one sub step.) More later!



	Disjunctions	
T I 1 01		

• If x is even, then $x^2 + x$ is even.



	Disjunctions ○○○●○○○○○○	

- If x is even, then $x^2 + x$ is even.
- If x is odd, then $x^2 + x$ is even.



	Disjunctions	

- If x is even, then $x^2 + x$ is even.
- If x is odd, then $x^2 + x$ is even.
- x is either odd or even.¹



- If x is even, then $x^2 + x$ is even.
- If x is odd, then $x^2 + x$ is even.
- x is either odd or even.¹
- Thus, $x^2 + x$ is even.



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- We call this the *constructive dilemma*: From only knowing the conclusion, we can't know which of the cases applied!



- If x is even, then $x^2 + x$ is even.
- If x is odd, then $x^2 + x$ is even.
- x is either odd or even.¹
- Thus, $x^2 + x$ is even.
- We call this the *constructive dilemma*: From only knowing the conclusion, we can't know which of the cases applied!
- Formally, this can be expressed as $p
 ightarrow r, q
 ightarrow r, p \lor q dash r$



The 1-Step Rules: Disjunction-Elimination Rule

Disjunction-Elimination Rule:

- This does not help us so much:
 - It's too restrictive because it requires implications to work! (Which would get eliminated as well.)
 - But we only want to eliminate the disjunction without further restrictions on the rest!



The 1-Step Rules: Disjunction-Elimination Rule

Disjunction-Elimination Rule:

- This does not help us so much:
 - It's too restrictive because it requires implications to work! (Which would get eliminated as well.)
 - But we only want to eliminate the disjunction without further restrictions on the rest!
- So, what do we do?





The 1-Step Rules: Disjunction-Elimination Rule (Based on Sequents)

 Deduction equivalence: X ⊢ A → B iff X, A ⊢ B Thus, we can re-write the previous rule as follows:



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 Deduction equivalence: X ⊢ A → B iff X, A ⊢ B Thus, we can re-write the previous rule as follows:

- Now we:
 - ... don't rely on implications anymore!
 - ... can discharge two assumptions (*A* and *B*), i.e., exactly those of the disjunction (but from two different sequents!).



The 1-Step Rules: Disjunction-Elimination Rule (Based on Sequents)

 Deduction equivalence: X ⊢ A → B iff X, A ⊢ B Thus, we can re-write the previous rule as follows:

- Now we:
 - ... don't rely on implications anymore!
 - ... can discharge two assumptions (*A* and *B*), i.e., exactly those of the disjunction (but from two different sequents!).
- Some good news and bad news: This is the hardest rule in natural deduction (So practice it!)



Introduction Negation Disjunctions Summary

The 1-Step Rules: When to Use that Rule

$$\frac{X \vdash A \lor B \quad Y, A \vdash C \quad Z, B \vdash C}{X, Y, Z \vdash C} \lor E$$

• Technically, this rule is used to "eliminate" a disjunction.





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Introduction Negation Disjunctions Summary

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- I.e., when we want to derive a disjunction, we can use it as C but this will also require another disjunction for the first sequent!



Introduction Negation Disjunctions Summary

$$\frac{X \vdash A \lor B \quad Y, A \vdash C \quad Z, B \vdash C}{X, Y, Z \vdash C} \lor E$$

- Technically, this rule is used to "eliminate" a disjunction.
- But in practice, we use it to prove one!
- How is that possible? Because we can use any formula for C!
- I.e., when we want to derive a disjunction, we can use it as C but this will also require another disjunction for the first sequent!
- We often obtain that one via assuming it.





• Disjunction is commutative: $p \lor q \vdash q \lor p$

 $lpha_1$ (1) $p \lor q$ A



• Disjunction is commutative: $p \lor q \vdash q \lor p$

 $\begin{array}{cccc} \alpha_1 & (1) & p \lor q & \mathsf{A} \\ \alpha_2 & (2) & p & \mathsf{A} \end{array}$





• Disjunction is commutative: $p \lor q \vdash q \lor p$

$$\begin{array}{cccc} \alpha_1 & (1) & p \lor q & \mathsf{A} \\ \alpha_2 & (2) & p & \mathsf{A} \\ \alpha_2 & (3) & q \lor p & 2 \lor I \end{array}$$





• Disjunction is commutative: $p \lor q \vdash q \lor p$

$$\begin{array}{ccccc} \alpha_1 & (1) & p \lor q & \mathsf{A} \\ \alpha_2 & (2) & p & \mathsf{A} \\ \alpha_2 & (3) & q \lor p & 2 \lor I \\ \alpha_3 & (4) & q & \mathsf{A} \end{array}$$







• Disjunction is commutative: $p \lor q \vdash q \lor p$



$$\boxed{\frac{X\vdash A}{X\vdash A\lor B}\lor l}$$

$$\frac{X \vdash A \lor B \quad Y, A \vdash C \quad Z, B \vdash C}{X, Y, Z \vdash C} \lor E$$



• Disjunction is commutative: $p \lor q \vdash q \lor p$

 α_1 (1) $p \lor q$ Α (2) А α_2 р (3) $q \lor p$ 2 ∨*I* α_2 А (4) α_3 q (5) $q \lor p$ 4 *∨I* α_3 $q \lor p$ 1,3[α_2],5[α_3] $\lor E$ (6) α_1





$$\begin{array}{c|cccc} \hline X \vdash A \lor B & Y, A \vdash C & Z, B \vdash C \\ \hline X, Y, Z \vdash C & \\ \hline & Z = \emptyset & \\ \hline & Z = \emptyset & \\ \hline & C = q \lor p \end{array}$$



• Disjunction is commutative: $p \lor q \vdash q \lor p$

 $\boxed{\frac{X \vdash B}{X \vdash A \lor B} \lor I}$



- In line 3, the q was just some arbitrary truth value that we've added due to Disjunction-Introduction!
- Similarly in line 5 the *p* was arbitrary. Notably, that's not the *p* from assumption α₂.



Conjunction and disjunction behave just like multiplication and addition, e.g. p ⋅ (q + r) = p ⋅ q + p ⋅ r:

 $p, q \lor r \vdash (p \land q) \lor (p \land r)$



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 p, q ∨ r ⊢ (p ∧ q) ∨ (p ∧ r)

Just a comment:

The analogy would have been stronger if instead of using the two assumptions $\alpha_1 = p$ and $\alpha_2 = q \lor r$, only a single assumption $\alpha'_1 = p \land (q \lor r)$ would have been used. (You can prove the other on your own.)



$$\alpha_1$$
 (1) p A



α_1	(1)	р	Α
α_2	(2)	$q \lor r$	Α



$$\begin{array}{cccc} \alpha_1 & (1) & p & A \\ \alpha_2 & (2) & q \lor r & A \end{array} \boxed{\begin{array}{cccc} X \vdash A \lor B & Y, A \vdash C & Z, B \vdash C \\ \hline & X, Y, Z \vdash C \end{array}} \lor E$$

$$\alpha_1, \alpha_2$$
 (n) $(p \land q) \lor (p \land r)$ x,y[?],z[?] $\lor E$



$$\alpha_1, \alpha_2$$
 (n) $(p \land q) \lor (p \land r)$ x,y $[\alpha_3], z[\alpha_4] \lor E$



Conjunction and disjunction behave just like multiplication and addition, e.g. p · (q + r) = p · q + p · r:
 p, q ∨ r ⊢ (p ∧ q) ∨ (p ∧ r)

$lpha_1$ $lpha_2$ $lpha_3$	(1) (2) (3)	$q \lor r$	A A A		$\frac{Y, A \vdash C}{X, Y, Z \vdash C}$	$Z, B \vdash C$
α_3 α_4 α_1, α_3	(4)	r	,,	A 1,3 ∧	I	$\boxed{\frac{X \vdash B}{X \vdash A \lor B} \lor I}$

 α_1, α_2 (n) $(p \land q) \lor (p \land r)$ x,y[α_3],z[α_4] $\lor E$



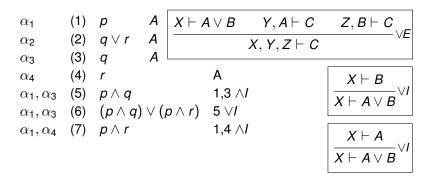
Conjunction and disjunction behave just like multiplication and addition, e.g. p · (q + r) = p · q + p · r:
 p, q ∨ r ⊢ (p ∧ q) ∨ (p ∧ r)

α_1 α_2	(1) (2)	p $Aq \lor r A$	$X \vdash A$		$\frac{Y, A \vdash C}{X, Y, Z \vdash C}$	$Z, B \vdash C$
α_3	(3)	q A			$\lambda, I, Z \vdash C$	
α_4	(4)			A		$X \vdash B$
α_1, α_3	(5)	$p \wedge q$		1,3 ∧/		$\left \frac{X \vdash A \lor B}{X \vdash A \lor B} \right $
α_1, α_3	(6)	$(p \land q) \lor (p$	v∧r)	5 ∨ <i>I</i>		

 α_1, α_2 (n) $(p \land q) \lor (p \land r)$ x,y[α_3],z[α_4] $\lor E$



Conjunction and disjunction behave just like multiplication and addition, e.g. p · (q + r) = p · q + p · r:
 p, q ∨ r ⊢ (p ∧ q) ∨ (p ∧ r)



 α_1, α_2 (n) $(p \land q) \lor (p \land r)$ x,y $[\alpha_3], z[\alpha_4] \lor E$



Conjunction and disjunction behave just like multiplication and addition, e.g. p · (q + r) = p · q + p · r:
 p, q ∨ r ⊢ (p ∧ q) ∨ (p ∧ r)

α_1	(1)	p	Α	$X \vdash A$	V B	$Y, A \vdash C$	$Z, B \vdash C$
α_2	(2)	$q \lor r$	Α			$X, Y, Z \vdash C$	
$lpha_{3}$	(3)	q	Α			, ,	
$lpha_{4}$	(4)	r			А		$X \vdash B$
α_1, α_3	(5)	$p \wedge q$			1,3 ∧/	1	$\left {X \vdash A \lor B} \lor I \right $
α_1, α_3	(6)	$(p \land q)$	\vee ()	$o \wedge r)$	5 ∨ <i>I</i>		
α_1, α_4	(7)	$p \wedge r$			1,4 ∧/	1	$X \vdash A$
α_1, α_4	(8)	$(p \land q)$	\vee ($o \wedge r)$	7 ∨ <i>I</i>		$\left \frac{X + Y}{X + A \vee B} \vee I \right $

 α_1, α_2 (n) $(p \land q) \lor (p \land r)$ x,y $[\alpha_3], z[\alpha_4] \lor E$



Conjunction and disjunction behave just like multiplication and addition, e.g. p · (q + r) = p · q + p · r:
 p, q ∨ r ⊢ (p ∧ q) ∨ (p ∧ r)

$lpha_1$ $lpha_2$ $lpha_3$	(1) (2) (3)	$\begin{array}{ccc} p & A \\ q \lor r & A \\ q & A \end{array}$	$\boxed{X \vdash A \lor E}$	$\begin{array}{ccc} B & Y, A \vdash C \\ \hline X, Y, Z \vdash C \end{array}$	$Z, B \vdash C$
α_4 α_1, α_3 α_1, α_3	(4) (5)	r $p \land q$ $(p \land q) \lor (p \land q)$	A 1,3 p∧r) 5∨		$\frac{X \vdash B}{X \vdash A \lor B} \lor I$
$ \begin{array}{c} \alpha_1, \alpha_4 \\ \alpha_1, \alpha_4 \\ \alpha_1, \alpha_2 \end{array} $	(7) (8)	$p \wedge r$ $(p \wedge q) \lor (p$	1,4 p∧r) 7∨	$\wedge I$	$\boxed{\frac{X \vdash A}{X \vdash A \lor B} \lor I}$

 α_1, α_2 (n) $(p \land q) \lor (p \land r)$ x,y $[\alpha_3]$,z $[\alpha_4] \lor E$



 $p
ightarrow r, q
ightarrow s dash (p \lor q)
ightarrow (r \lor s)$



Introduction	Negation		Disjunctions ○○○○○○○○●	Summary 00
The 1-Step Rules:	Disjunction-Intro	duction and -El	imination, Example 3	3
ho ightarrow	r,q ightarrow s dash (p)	$\lor q) ightarrow (r \lor s)$)	
α_1	(1)	ho ightarrow r	А	

 α_1 (1) $p \rightarrow r$

$$\alpha_2$$
 (2) $q \rightarrow s$ A



Introduction	Negation	0	Disjunctions ○○○○○○○○●	Summary 00
The 1-Step Rules:	Disjunction-In	troduction and -E	limination, Example	3
ho ightarrow	$r, q \rightarrow s \vdash ($	$p \lor q) ightarrow (r \lor q)$	<i>s</i>)	
α_1	(1)	ho ightarrow r	А	

$$\alpha_2$$
 (2) $q \rightarrow s$ A

$$\alpha_1, \alpha_2$$
 (n) $(p \lor q) \to (r \lor s)$



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coccoccoccoccoccoccoThe 1-Step Rules:Disjunction-Introduction and -Elimination, Example 3 $p \rightarrow r, q \rightarrow s \vdash (p \lor q) \rightarrow (r \lor s)$ α_1 (1) α_2 (2) $q \rightarrow s$ A α_3 (3) $p \lor q$ A

$$\begin{array}{lll} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-1}) & r \lor s \\ \alpha_1, \alpha_2 & (\mathsf{n}) & (p \lor q) \to (r \lor s) & (\mathsf{n-1})[\alpha_3] \to I \end{array}$$



Introduction
coNegation
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coccoccoccoccoSummary
coccoccoccoccoThe 1-Step Rules:Disjunction-Introduction and -Elimination, Example 3 $p \rightarrow r, q \rightarrow s \vdash (p \lor q) \rightarrow (r \lor s)$ α_1 (1) $p \rightarrow r$ α_2 (2) $q \rightarrow s$ α_3 (3) $p \lor q$

$$\frac{X \vdash A \lor B \quad Y, A \vdash C \quad Z, B \vdash C}{X, Y, Z \vdash C} \lor E$$

 $\begin{array}{cccc} \alpha_1, \alpha_2, \alpha_3 & (\text{n-1}) & r \lor s & x, y[?], z[?] \lor E \\ \alpha_1, \alpha_2 & (\text{n}) & (p \lor q) \to (r \lor s) & (\text{n-1})[\alpha_3] \to I \end{array}$



Disjunctions 000000000 The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 3 $p \rightarrow r, q \rightarrow s \vdash (p \lor q) \rightarrow (r \lor s)$ (1) $p \rightarrow r$ А α_1 (2) $q \rightarrow s$ А as (3) $p \lor q$ А α_3 (4) р А α_{4} (5) А α_5 q $X \vdash A \lor B$ $Y, A \vdash C$ $Z, B \vdash C$ $\lor E$ $X = \{ p \lor q \} \quad A = \alpha_4 = p$ Y = ? $B = \alpha_5 = q$ $X, Y, Z \vdash C$ Z = ? $C = r \lor s$

 $\begin{array}{cccc} \alpha_1, \alpha_2, \alpha_3 & (n-1) & r \lor s & 3, y[\alpha_4], z[\alpha_5] \lor E \\ \alpha_1, \alpha_2 & (n) & (p \lor q) \to (r \lor s) & (n-1)[\alpha_3] \to I \end{array}$

Disjunctions 000000000 The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 3 $p \rightarrow r, q \rightarrow s \vdash (p \lor q) \rightarrow (r \lor s)$ (1) $p \rightarrow r$ А α_1 (2) $q \rightarrow s$ А α_2 (3) $p \lor q$ А α_3 (4) р А α_{4} (5) А α_5 q $1.4 \rightarrow E$ (6)r α_1, α_4

 $\begin{array}{lll} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-1}) & r \lor s & & \mathbf{3}, \mathsf{y}[\alpha_4], \mathsf{z}[\alpha_5] \lor \mathsf{E} \\ \alpha_1, \alpha_2 & (\mathsf{n}) & (p \lor q) \to (r \lor s) & (\mathsf{n-1})[\alpha_3] \to \mathsf{I} \end{array}$



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 $\begin{array}{lll} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-1}) & r \lor s & & \mathbf{3}, \mathsf{y}[\alpha_4], \mathsf{z}[\alpha_5] \lor \mathsf{E} \\ \alpha_1, \alpha_2 & (\mathsf{n}) & (p \lor q) \to (r \lor s) & (\mathsf{n-1})[\alpha_3] \to \mathsf{I} \end{array}$



Introduction 00	Negation		Disjunctions ○○○○○○○○●	Summary 00
The 1-Step Rules:	Disjunction-Intr	roduction and -Eli	mination, Example 3	
p –	ightarrow r, q ightarrow s dash (p	$(r \lor q) \to (r \lor s)$)	
α_1	(1)	p ightarrow r	А	
α_2	(2)	$oldsymbol{q} ightarrow oldsymbol{s}$	A	
$lpha_{3}$	(3)	$p \lor q$	A	
$lpha_{4}$	(4)	p	A	
α_5	(5)	q	А	
α_1 ,	<i>α</i> ₄ (6)	r	1,4 $ ightarrow E$	
$\alpha_1,$	<i>α</i> ₄ (7)	$r \lor s$	6 <i>∨I</i>	
$\alpha_2,$	<i>α</i> ₅ (8)	S	2,5 ightarrow E	

 $\begin{array}{lll} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-1}) & r \lor s & & \mathsf{3,y}[\alpha_4], \mathsf{z}[\alpha_5] \lor \mathsf{E} \\ \alpha_1, \alpha_2 & (\mathsf{n}) & (p \lor q) \to (r \lor s) & (\mathsf{n-1})[\alpha_3] \to \mathsf{I} \end{array}$



Introduction 00	Negation	0000000		Disjunctions ○○○○○○○○●	Summary 00
The 1-Step F	lules: Disjunct	ion-Intr	oduction and -El	imination, Example 3	
	ho ightarrow r, q ightarrow	$s \vdash (p$	$(r \lor q) \to (r \lor s)$	5)	
	α_1	(1)	ho ightarrow r	А	
	α_2	(2)	q ightarrow s	А	
	α_3	(3)	$p \lor q$	А	
	α_4	(4)	p	А	
	α_5	(5)	q	А	
	α_1, α_4	(6)	r	1,4 $ ightarrow E$	
	α_1, α_4	(7)	$r \lor s$	6 <i>∨I</i>	
	α_2, α_5	(8)	S	2,5 $ ightarrow E$	
	α_2, α_5	(9)	$r \lor s$	8 <i>∨I</i>	

 $\begin{array}{cccc} \alpha_1, \alpha_2, \alpha_3 & (\mathsf{n-1}) & r \lor s & & \mathbf{3}, \mathbf{y}[\alpha_4], \mathbf{z}[\alpha_5] \lor E \\ \alpha_1, \alpha_2 & (\mathsf{n}) & (p \lor q) \to (r \lor s) & (\mathsf{n-1})[\alpha_3] \to I \end{array}$



Disjunctions 000000000 The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 3 $p \rightarrow r, q \rightarrow s \vdash (p \lor q) \rightarrow (r \lor s)$ (1) А α_1 $p \rightarrow r$ (2) $q \rightarrow s$ А as (3) А $p \lor q$ α_3 (4) р А α_{4} (5) А α_5 q $1.4 \rightarrow E$ (6) r α_1, α_4 6 ∨*I* (7) $r \lor s$ α_1, α_4 $2.5 \rightarrow E$ (8) s α_2, α_5 (9) $r \vee s$ 8 VI α_2, α_5 3,7[α_4],9[α_5] $\forall E$ (10) $r \lor s$ $\alpha_1, \alpha_2, \alpha_3$ $3,y[\alpha_4],z[\alpha_5] \lor E$ (n-1) *r* ∨ *s* $\alpha_1, \alpha_2, \alpha_3$ (n) $(p \lor q) \to (r \lor s)$ $(n-1)[\alpha_3] \to I$ α_1, α_2



ntroduction	Negatio 00000	n 000000000	Disjuncti		Summary 00
The 1-Step	Rules: Disjund	ction-Int	roduction and -Elimina	tion, Example 3	
	p ightarrow r, q -	∍ s ⊢ (µ	$p \lor q) ightarrow (r \lor s)$		
	α_1	(1)	ho ightarrow r	А	
	α_2	(2)	q ightarrow s	А	
	$lpha_{3}$	(3)	$p \lor q$	А	
	α_4	(4)	p	Α	
	α_5	(5)	q	Α	
	α_1, α_4	(6)	r	1,4 $ ightarrow E$	
	α_1, α_4	(7)	$r \lor s$	6 <i>∨I</i>	
	α_2, α_5	(8)	S	2,5 $ ightarrow E$	
	α_{2}, α_{5}	(9)	$r \lor s$	8 <i>∨I</i>	
	$\alpha_1, \alpha_2, \alpha_3$. ,	$r \lor s$	3 ,7[α ₄],9[α ₅] ∨ <i>E</i>	
	α_1, α_2	(11)	$(p \lor q) ightarrow (r \lor s)$	10[α ₃] →/	
	$\alpha_1, \alpha_2, \alpha_3$	(n-1)		3,y[α_4],z[α_5] ∨ <i>E</i>	
	α_1, α_2	(n)	$(p \lor q) ightarrow (r \lor s)$	$(n-1)[\alpha_3] \rightarrow I$	



Summary



• The remaining rules for natural deduction: negation and disjunction



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- Note that, this time, we had more than just 1-step rules!
- ightarrow The entire Logic Notes sections:
 - Propositional natural deduction: Negation
 - Propositional natural deduction: Disjunction
 - $\rightarrow~$ We are done now with everything until Section 2!

