Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Natural Deduction — Negation, Disjunction

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Introduction

Recap on Natural Deduction

- What are theorems? (Sequents without assumptions!)
- Relationship between $\vdash$ and $\rightarrow$:
  - They live in completely different worlds!
  - $\rightarrow$ is a connective and thus part of a formula, just like $\neg$, $\wedge$, and $\vee$.
  - $\vdash$ is *not* a connective and can thus not possibly be part of *any* formula! It only states whether we can derive a single formula A from a set of formulae X, expressed by $X \vdash A$.
- How do proofs in natural deduction look?
  - We use a list/table format with 4 columns.
  - All of these columns are *essential*!
- Introduction and elimination rules for:
  - Conjunction (easy!)
  - Implication (not quite that easy!)
- So what's missing?
  - Negation (not as easy as you might think!)
  - Disjunction (*quite* hard... Practice it!)

Negation

Introduction: Intuitive Meaning

- What does the negation connective in logics mean?
- It inverts truth values! Remember our introductory example:
  - Socrates is a goat ($= p$)
  - It's not true that Socrates is a goat ($= \neg p$)
- Be careful when translating "not" used in natural language:
  - Someone likes Logic ($= p$)
  - Someone doesn't like Logic! ($\neq \neg p$)
  - Such complex propositions will be covered in *predicate logic*!

Introduction: Truth Table

- Since the not connective simply inverts a single truth value we get a simple truth table:

p	$\neg p$
0	1
1	0

p	$\neg p$	$\neg\neg p$
0	1	0
1	0	1

- I.e., in propositional logic, two negations eliminate each other!
- It's not true that it's not true that Socrates is a goat (So it *is* true!)

The 1-Step Rules: Double-Negation Elimination and Introduction

- The (second) truth table gives us the following two rules:
- **Double-Negation Elimination and Introduction Rules:**

$$\frac{\neg\neg A}{A}\,\neg\neg E \qquad \frac{A}{\neg\neg A}\,\neg\neg I$$

- Again based on sequents:

$$\frac{X \vdash \neg\neg A}{X \vdash A}\,\neg\neg E \qquad \frac{X \vdash A}{X \vdash \neg\neg A}\,\neg\neg I$$

The 1-Step Rules: A Mistake That Will Cost You Marks

Avoid the next common mistake:

- Look carefully what/where the main connective is!
- The rule refers to a *complete formula*!
- So, e.g., we cannot go from $p \wedge \neg\neg q$ to $p \wedge q$ in just one step!

$$\frac{\neg\neg A}{A}\,\neg\neg E \qquad \frac{A}{\neg\neg A}\,\neg\neg I$$

Because $A = (p \wedge \neg\neg q)$,
but the rule states it should be $\neg\neg q$!

The 1-Step Rules: A Mistake That Will Cost You Marks

Avoid the next common mistake:

- Look carefully what/where the main connective is!
- The rule refers to a *complete formula*!
- So, e.g., we cannot go from $p \wedge \neg\neg q$ to $p \wedge q$ in just one step!
- Correctly handling that: (with a slightly more complex example)

$$p \wedge \neg\neg q \vdash \neg\neg p \wedge q$$

α_1	(1)	$p \wedge \neg\neg q$	A
α_1	(2)	p	1 $\wedge E$
α_1	(3)	$\neg\neg p$	2 $\neg\neg I$
α_1	(4)	$\neg\neg q$	1 $\wedge E$
α_1	(5)	q	4 $\neg\neg E$
α_1	(6)	$\neg\neg p \wedge q$	3,5 $\wedge I$

The 1-Step Rules: Negation-Elimination

- With the double-negation rules we can't introduce or eliminate a *single* negation.
- To deal with single negations, we require the symbol $\bot$.
- We introduced it before: it represents "false", an "absurd" constant that can never be satisfied.
- **Negation-Elimination rule:** (without and with sequent-notation)

$$\frac{A \qquad \neg A}{\bot}\neg E \qquad\qquad \frac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\neg E$$

The 1-Step Rules: Negation-Introduction

- **Negation-Introduction rule:** (without and with sequent-notation)

$$\frac{\begin{matrix}[A] \\ \vdots \\ \bot\end{matrix}}{\neg A}\neg I \qquad\qquad \frac{X, A \vdash \bot}{X \vdash \neg A}\neg I$$

- Negation-Introduction discharges assumption A.
- Interesting fact(s):
  - Since we do not pose further restrictions on A, we can blame the contradiction on anything we want! E.g., if $X = \{A_1, \ldots, A_n\}$ and $X \vdash \bot$, we can conclude $X \setminus \{A_i\} \vdash \neg A_i$ for any $A_i \in X$.
  - This rule is the main proof idea behind the proof technique *"Proof by contradiction"*. (There are, e.g., nice illustrations on YouTube proving that $\sqrt{2}$ is not rational by that technique.)

The 1-Step Rules: Excursion, Proof by Contradiction

- We want to show:

  If you are in Canberra (p), you are $\overbrace{\text{not in Sydney}(\neg q)}^{\text{You are in Sydney}(q)}$; thus:
  if you are in Sydney (q), you are not in Canberra ($\neg p$)
  I.e., $p \rightarrow \neg q \vdash q \rightarrow \neg p$
- Proof by contradiction:
  - Assume the premise (i.e., $p \rightarrow \neg q$) is true and *additionally* make
  - the assumption that the conclusion (i.e., $q \rightarrow \neg p$) is fase!
  - Thus, "You are in Sydney *and* in Canberra", $I(q) = 1$ and $I(p) = 1$

Why?

p	q	$\neg p$	$q \rightarrow \neg p$	$\neg(q \rightarrow \neg p)$
0	0	1	1	0
0	1	1	1	0
1	0	0	1	0
1	1	0	0	1

The 1-Step Rules: Excursion, Proof by Contradiction

- We want to show:

  If you are in Canberra (p), you are $\underbrace{\text{not in Sydney}}^{\text{You are in Sydney}(q)}(\neg q)$; thus:
  if you are in Sydney (q), you are not in Canberra ($\neg p$)
  I.e., $p \rightarrow \neg q \vdash q \rightarrow \neg p$
- Proof by contradiction:
  - Assume the premise (i.e., $p \rightarrow \neg q$) is true and *additionally* make
  - the assumption that the conclusion (i.e., $q \rightarrow \neg p$) is fase!
  - Thus, “You are in Sydney *and* in Canberra”, $I(q) = 1$ and $I(p) = 1$
  - Because of the first assumption, and since we just assumed we are in Canberra, we can conclude that we are not in Sydney.
  - But now we are in Sydney, and not in Sydney, contradiction!
  - Thus our additional assumption that the second implication is false must be wrong, so it must be true!
  - Thus, the first implication implies the second! q.e.d.

The 1-Step Rules: Negation-Elimination and -Introduction, Example 1

- If you are in Canberra (p), you are not in Sydney ($\neg q$); thus:
  if you are in Sydney (q), you are not in Canberra ($\neg p$)

$p \rightarrow \neg q \vdash q \rightarrow \neg p$

α_1	(1)	$p \rightarrow \neg q$	A
α_2	(2)	q	A
α_3	(3)	p	A
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$\bot$	x,y $\neg E$
α_1, α_2	(n-1)	$\neg p$	(n-2)[α_3] $\neg I$
α_1	(n)	$q \rightarrow \neg p$	(n-1) $\rightarrow I$

$$\frac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\neg E$$

$$\frac{X, A \vdash \bot}{X \vdash \neg A}\neg I$$

The 1-Step Rules: Negation-Elimination and -Introduction, Example 1

- If you are in Canberra (p), you are not in Sydney ($\neg q$); thus:
  if you are in Sydney (q), you are not in Canberra ($\neg p$)

$p \rightarrow \neg q \vdash q \rightarrow \neg p$

α_1	(1)	$p \rightarrow \neg q$	A
α_2	(2)	q	A
α_3	(3)	p	A
α_1, α_3	(4)	$\neg q$	1,3 $\rightarrow E$
$\alpha_1, \alpha_2, \alpha_3$	(5)	$\bot$	2,4 $\neg E$
α_1, α_2	(6)	$\neg p$	5[α_3] $\neg I$
α_1	(7)	$q \rightarrow \neg p$	6[α_2] $\rightarrow I$
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	$\bot$	x,y $\neg E$
α_1, α_2	(n-1)	$\neg p$	(n-2)[α_3] $\neg I$
α_1	(n)	$q \rightarrow \neg p$	(n-1) $\rightarrow I$

$$\frac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\neg E$$

$$\frac{X, A \vdash \bot}{X \vdash \neg A}\neg I$$

The 1-Step Rules: Negation-Elimination and -Introduction, Example 2

- Contradict yourself, and I don’t care anymore!
- In other words: We can conclude all we want from an inconsistent knowledge base.

$p, \neg p \vdash q$

α_1	(1)	p	A
α_2	(2)	$\neg p$	A
α_1, α_2	(3)	$\bot$	1,2 $\neg E$
α_1, α_2	(4)	$\neg\neg q$	3[] $\neg I$
α_1, α_2	(5)	q	4 $\neg\neg E$

$$\frac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\neg E$$

$$\frac{X, A \vdash \bot}{X \vdash \neg A}\neg I \qquad \frac{X \vdash \neg\neg A}{X \vdash A}\neg\neg E$$

- Here we have another example of vacuous discharge: We blame the contradiction on a non-existing assumption $\neg q$.

A 2-Step Rule: Reductio ad Absurdum (RAA)

- We can combine Negation-Elimination with its Introduction: Again, notations without and with sequents:

$$\frac{\begin{matrix}[B] & & [B]\\ \vdots & & \vdots \\ A & & \neg A\end{matrix}}{\neg B}\,RAA \qquad \frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\,RAA$$

- The rules discharge assumption B.
- Why is it correct?

$$\frac{X, A \vdash \bot}{X \vdash \neg A}\,\neg I \qquad \frac{X \vdash A \qquad Y \vdash \neg A}{X, Y \vdash \bot}\,\neg E \qquad \frac{\dfrac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y, B \vdash \bot}\,\neg E}{X, Y \vdash \neg B}\,\neg I$$

A 2-Step Rule: Reductio ad Absurdum (RAA), Example 1

- $p \to \neg p \vdash \neg p$: p is *so false*, it implies its own negation!

  Or: Since p and $\neg p$ can't be true at the same time, the implication $p \to \neg p$ cannot be "activated", so its precondition must be false.

$p \to \neg p \vdash \neg p$

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\,RAA$$

α_1	(1)	$p \to \neg p$	A
α_2	(2)	p	A
α_1, α_2	(3)	$\neg p$	1,2 $\to E$
α_1	(4)	$\neg p$	2,3[α_2] *RAA*
α_1	(n)	$\neg p$	x,y[α_2] *RAA*

A 2-Step Rule: Reductio ad Absurdum (RAA), Example 2

- $\neg p \to p \vdash p$: if p is even implied by its own negation, then it *must* be true!

  Again! Since p and $\neg p$ can't be true at the same time, the implication $\neg p \to p$ cannot be "activated", so its precondition must be false.

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\,RAA$$

$\neg p \to p \vdash p$

α_1	(1)	$\neg p \to p$	A
α_2	(2)	$\neg p$	A
α_1, α_2	(3)	p	1,2 $\to E$
α_1	(4)	$\neg\neg p$	2,3[α_2] *RAA*
α_1	(5)	p	4 $\neg\neg E$

Disjunctions

Introduction: (Our) Or versus Exclusive Or

- Disjunctions are of the form $A \vee B$
- It rains this afternoon or this evening.
  (But it can also be both!)
- The cat is *either* dead *or* alive.
  (Unless it's a physicist's cat, the choice is *exclusive*!
  The cat *cannot be both dead and alive*!)
- We use the first, non-exclusive, notion of or:
  At least one proposition needs to be true!

The 1-Step Rules: Disjunction-Introduction

Disjunction-Introduction Rules:

- Notation without sequents:

$$\frac{A}{A \vee B}\vee I \qquad \frac{B}{A \vee B}\vee I$$

- Notation with sequents:

$$\frac{X \vdash A}{X \vdash A \vee B}\vee I \qquad \frac{X \vdash B}{X \vdash A \vee B}\vee I$$

- Great! So we have that easy rule to prove a disjunction, right?
- Well... No. (That's only one sub step.) More later!

The 1-Step Rules: Disjunction-Elimination, Introduction

- If x is even, then $x^2 + x$ is even.
- If x is odd, then $x^2 + x$ is even.
- x is either odd or even.[1]
- Thus, $x^2 + x$ is even.
- We call this the *constructive dilemma*: From only knowing the conclusion, we can't know which of the cases applied!
- Formally, this can be expressed as $p \to r, q \to r, p \vee q \vdash r$

[1] Technically, we use the exclusive or here, but the argument remains true even if it's the non-exclusive or.

The 1-Step Rules: Disjunction-Elimination Rule

- **Disjunction-Elimination Rule:**

$$\frac{A \vee B \qquad A \to C \qquad B \to C}{C}\vee E$$

- This does not help us so much:
  - It's too restrictive because it requires implications to work! (Which would get eliminated as well.)
  - But we only want to eliminate the disjunction without further restrictions on the rest!
- So, what do we do?

The 1-Step Rules: Disjunction-Elimination Rule (Based on Sequents)

- Deduction equivalence: $X \vdash A \rightarrow B$ iff $X, A \vdash B$
  Thus, we can re-write the previous rule as follows:

$$\dfrac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\vee E \qquad \dfrac{X \vdash A \vee B \quad Y, A \vdash C \quad Z, B \vdash C}{X, Y, Z \vdash C}\vee E$$

- Now we:
  - ... don't rely on implications anymore!
  - ... can discharge two assumptions (A and B), i.e., exactly those of the disjunction (but from two different sequents!).
- Some good news and bad news: This is the hardest rule in *natural deduction* (So practice it!)

The 1-Step Rules: When to Use that Rule

$$\dfrac{X \vdash A \vee B \quad Y, A \vdash C \quad Z, B \vdash C}{X, Y, Z \vdash C}\vee E$$

- Technically, this rule is used to "eliminate" a disjunction.
- But in practice, we use it to prove one!
- How is that possible? Because we can use *any* formula for C!
- I.e., when we want to derive a disjunction, we can use it as C – but this will also require another disjunction for the first sequent!
- We often obtain that one via assuming it.

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 1

- Disjunction is commutative: $p \vee q \vdash q \vee p$

α_1	(1)	$p \vee q$	A
α_2	(2)	p	A
α_2	(3)	$q \vee p$	2 $\vee I$
α_3	(4)	q	A
α_3	(5)	$q \vee p$	4 $\vee I$
α_1	(6)	$q \vee p$	1,3[α_2],5[α_3] $\vee E$

$$\dfrac{X \vdash B}{X \vdash A \vee B}\vee I \qquad \dfrac{X \vdash A}{X \vdash A \vee B}\vee I$$

$$\dfrac{X \vdash A \vee B \quad Y, A \vdash C \quad Z, B \vdash C}{X, Y, Z \vdash C}\vee E$$

$$X = \overbrace{\{p \vee q\}}^{\alpha_1} \quad A = \alpha_2 = p$$
$$Y = \emptyset \quad B = \alpha_3 = q$$
$$Z = \emptyset \quad C = q \vee p$$

Introduction Negation Disjunctions Summary

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 1

- Disjunction is commutative: $p \vee q \vdash q \vee p$

α_1	(1)	$p \vee q$	A
α_2	(2)	p	A
α_2	(3)	$q \vee p$	2 $\vee I$
α_3	(4)	q	A
α_3	(5)	$q \vee p$	4 $\vee I$
α_1	(6)	$q \vee p$	1,3[α_2],5[α_3] $\vee E$

$$\dfrac{X \vdash B}{X \vdash A \vee B}\vee I \qquad \dfrac{X \vdash A}{X \vdash A \vee B}\vee I$$

- In line 3, the q was just some *arbitrary* truth value that we've added due to Disjunction-Introduction!
- Similarly in line 5 the p was arbitrary. Notably, that's not the p from assumption α_2.

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 2

- Conjunction and disjunction behave just like multiplication and addition, e.g. $p \cdot (q + r) = p \cdot q + p \cdot r$:

$p, q \vee r \vdash (p \wedge q) \vee (p \wedge r)$

Just a comment:
The analogy would have been stronger
if instead of using the two assumptions
$\alpha_1 = p$ and $\alpha_2 = q \vee r$, only a single assumption
$\alpha_1' = p \wedge (q \vee r)$ would have been used.
(You can prove the other on your own.)

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 2

- Conjunction and disjunction behave just like multiplication and addition, e.g. $p \cdot (q + r) = p \cdot q + p \cdot r$:

$p, q \vee r \vdash (p \wedge q) \vee (p \wedge r)$

α_1	(1)	p	A
α_2	(2)	$q \vee r$	A
α_3	(3)	q	A
α_4	(4)	r	A
α_1, α_3	(5)	$p \wedge q$	1,3 $\wedge I$
α_1, α_3	(6)	$(p \wedge q) \vee (p \wedge r)$	5 $\vee I$
α_1, α_4	(7)	$p \wedge r$	1,4 $\wedge I$
α_1, α_4	(8)	$(p \wedge q) \vee (p \wedge r)$	7 $\vee I$
α_1, α_2	(9)	$(p \wedge q) \vee (p \wedge r)$	2,6[α_3],8[α_4] $\vee E$
α_1, α_2	(n)	$(p \wedge q) \vee (p \wedge r)$	x,y[α_3],z[α_4] $\vee E$

$$\frac{X \vdash A \vee B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C}\vee E$$

$$\frac{X \vdash B}{X \vdash A \vee B}\vee I$$

$$\frac{X \vdash A}{X \vdash A \vee B}\vee I$$

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 3

$p \rightarrow r, q \rightarrow s \vdash (p \vee q) \rightarrow (r \vee s)$

α_1	(1)	$p \rightarrow r$	A
α_2	(2)	$q \rightarrow s$	A
α_3	(3)	$p \vee q$	A
α_4	(4)	p	A
α_5	(5)	q	A
$\alpha_1, \alpha_2, \alpha_3$	(n-1)	$r \vee s$	3,y[α_4],z[α_5] $\vee E$
α_1, α_2	(n)	$(p \vee q) \rightarrow (r \vee s)$	(n-1)[α_3] $\rightarrow I$

$$\frac{X \vdash A \vee B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C}\vee E$$

$X = \{p \vee q\}$ $\quad A = \alpha_4 = p$
$Y = ?$ $\quad B = \alpha_5 = q$
$Z = ?$ $\quad C = r \vee s$

The 1-Step Rules: Disjunction-Introduction and -Elimination, Example 3

$p \rightarrow r, q \rightarrow s \vdash (p \vee q) \rightarrow (r \vee s)$

α_1	(1)	$p \rightarrow r$	A
α_2	(2)	$q \rightarrow s$	A
α_3	(3)	$p \vee q$	A
α_4	(4)	p	A
α_5	(5)	q	A
α_1, α_4	(6)	r	1,4 $\rightarrow E$
α_1, α_4	(7)	$r \vee s$	6 $\vee I$
α_2, α_5	(8)	s	2,5 $\rightarrow E$
α_2, α_5	(9)	$r \vee s$	8 $\vee I$
$\alpha_1, \alpha_2, \alpha_3$	(10)	$r \vee s$	3,7[α_4],9[α_5] $\vee E$
α_1, α_2	(11)	$(p \vee q) \rightarrow (r \vee s)$	10[α_3] $\rightarrow I$
$\alpha_1, \alpha_2, \alpha_3$	(n-1)	$r \vee s$	3,y[α_4],z[α_5] $\vee E$
α_1, α_2	(n)	$(p \vee q) \rightarrow (r \vee s)$	(n-1)[α_3] $\rightarrow I$

Summary

Content of this Lecture

- The remaining rules for natural deduction: negation and disjunction
- Note that, this time, we had more than just 1-step rules!

→ The entire Logic Notes sections:
  - Propositional natural deduction: Negation
  - Propositional natural deduction: Disjunction

  → We are done now with everything until Section 2!