Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: Propositional Logic
— Recap on Proof Strategies (and some System Support)

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Introduction

Recap: Sequents, Validity, Proof Systems

- You know how to prove $X \vdash A$ via ND
- You know how to prove $X \vdash A$ via ST
- You know that ND is only a semi-decision procedure, as you can only show $X \vdash A$ if that sequent is valid.
- So, to show *invalidity*, you have to rely on ST.

So, when to use which?

- In assignments/an exam:
  - We tell you what to use. :)
  - If we don't tell whether it's valid or invalid:
    - If you think the sequent is invalid: You must use ST
    - If you think the sequent is valid: Choose what you are stronger in!
    - If you don't know either way: Use ST and *let it tell you*!
- In real life:
  - Neither.'Use a SAT-solver like FINDER / Logic for Fun
  - Different solvers rely on different techniques, and there are competitions on identifying the quickest ones.

Strategies: Overview

Strategies for Semantic Tableau and Natural Deduction

Semantic Tableau:

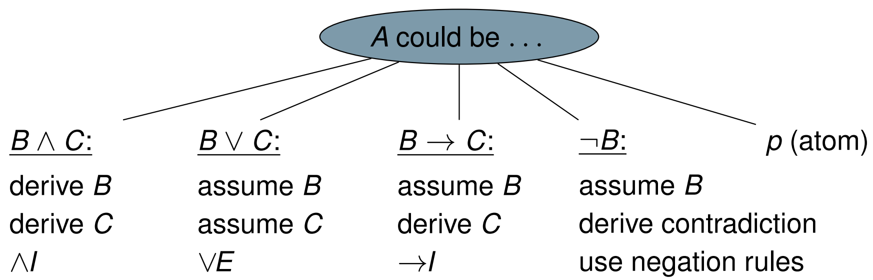
- Always apply rules first that don't branch.
- In case of invalid sequents you could follow down branches leading to an open branch more quickly – which requires "seeing" which interpretation proves invalidity.

Natural Deduction:

- Our standard strategy:
  - Write down all assumptions
  - Start with the sequent to prove as last line and apply rules until no more rule applies
  - Fill the gap!
- Fall-back strategy: Assume negation of final derivation and exploit contradiction.

Natural Deduction: Overview

How to show $X \vdash A$? Depends on A!



Note:

- $X \vdash A$ can also refer to sub steps!
- Usually, you will need $\vee E$ if $B \vee C \in X$, not if $B \vee C = A$.

Examples for Natural Deduction

$X \vdash A$, A is an Implication

$$p \rightarrow ((q \wedge r) \rightarrow s) \vdash q \rightarrow ((p \wedge r) \rightarrow s)$$

α_1	(1)	$p \rightarrow ((q \wedge r) \rightarrow s)$	A
α_2	(2)	q	A
α_3	(3)	$p \wedge r$	A
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	s	
α_1, α_2	(n-1)	$(p \wedge r) \rightarrow s$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$q \rightarrow ((p \wedge r) \rightarrow s)$	(n-1)[α_2] $\rightarrow I$

- Do *not* assume p! Use the strategies for A! (p is part of X)
- As always, we can perform regression to see where to go!
- Now we cannot go further via regression, so fill the gap!

$X \vdash A$, A is an Implication

$$p \rightarrow ((q \wedge r) \rightarrow s) \vdash q \rightarrow ((p \wedge r) \rightarrow s)$$

α_1	(1)	$p \rightarrow ((q \wedge r) \rightarrow s)$	A
α_2	(2)	q	A
α_3	(3)	$p \wedge r$	A
α_3	(4)	p	3 $\wedge E$
α_1, α_3	(5)	$(q \wedge r) \rightarrow s$	1,4 $\rightarrow E$
α_3	(6)	r	3 $\wedge E$
α_2, α_3	(7)	$q \wedge r$	2,6 $\wedge I$
$\alpha_1, \alpha_2, \alpha_3$	(8)	s	5,7 $\rightarrow E$
α_1, α_2	(9)	$(p \wedge r) \rightarrow s$	8[α_3] $\rightarrow I$
α_1	(10)	$q \rightarrow ((p \wedge r) \rightarrow s)$	9[α_2] $\rightarrow I$
$\alpha_1, \alpha_2, \alpha_3$	(n-2)	s	
α_1, α_2	(n-1)	$(p \wedge r) \rightarrow s$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$q \rightarrow ((p \wedge r) \rightarrow s)$	(n-1)[α_2] $\rightarrow I$

Tool Support (Illustrated on Previous Example)

$p \rightarrow ((q \wedge r) \rightarrow s) \vdash q \rightarrow ((p \wedge r) \rightarrow s)$

α_1	(1)	$p \rightarrow ((q \wedge r) \rightarrow s)$	A
α_2	(2)	q	A
α_3	(3)	$p \wedge r$	A
$\alpha_1, \alpha_2, \alpha_3$	(X)	s	
α_1, α_2	(Y)	$(p \wedge q) \rightarrow s$	
α_1	(Z)	$q \rightarrow ((p \wedge r) \rightarrow s)$	Y [α_2]

According to the previous slide, the last three lines should be:

$\alpha_1, \alpha_2, \alpha_3$	(n-2)	s	
α_1, α_2	(n-1)	$(p \wedge r) \rightarrow s$	(n-2)[α_3] $\rightarrow I$
α_1	(n)	$q \rightarrow ((p \wedge r) \rightarrow s)$	(n-1)[α_2] $\rightarrow I$

Annotating goal

a1 | q → ((p ∧ r) → s) | Y[a2] | →I
Rule used | CANCEL

But when the rule application for line Z is added, the checker finds a problem, see below.

The premise of →I must be the consequent of the derived conditional.
In this case, the consequent is $(p \wedge r) \rightarrow s$ but the premise on line Y is $(p \wedge q) \rightarrow s$

Choose from the menu on the right:

--- select ---

Tool Support (Illustrated on Previous Example), cont'd

The corrected version looks like:

$p \rightarrow ((q \wedge r) \rightarrow s) \vdash q \rightarrow ((p \wedge r) \rightarrow s)$

α_1	(1)	$p \rightarrow ((q \wedge r) \rightarrow s)$	A
α_2	(2)	q	A
α_3	(3)	$p \wedge r$	A
$\alpha_1, \alpha_2, \alpha_3$	(X)	s	
α_1, α_2	(Y)	$(p \wedge r) \rightarrow s$	X [α_3] $\rightarrow$I
α_1	(Z)	$q \rightarrow ((p \wedge r) \rightarrow s)$	Y [α_2] $\rightarrow$I

Next action

- All problems will be detected by the checker!
- You find the link to the checker (and its manual) on Wattle.

$X \vdash A$, A is a Negation

$\neg(p \vee q) \vdash \neg p$

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

α_1	(1)	$\neg(p \vee q)$	A
α_2	(2)	p	A
α_2	(3)	$p \vee q$	2 $\vee I$
α_1	(4)	$\neg p$	1,3[α_2] RAA
α_1	(n)	$\neg p$	x,y[α_2] RAA

$X \vdash A$, A is a Disjunction (here: in one of the Substeps)

$\neg p \wedge \neg q \vdash \neg(p \vee q)$

$$\frac{X \vdash A \vee B \qquad Y, A \vdash C \qquad Z, B \vdash C}{X, Y, Z \vdash C} \vee E$$

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

α_1	(1)	$\neg p \wedge \neg q$	A
α_2	(2)	$p \vee q$	A
α_3	(3)	p	A
α_4	(4)	q	A
α_1	(5)	$\neg p$	1 $\wedge E$
α_3	(6)	$\neg(\neg p \wedge \neg q)$	3,5[α_1] RAA
α_1	(7)	$\neg q$	1 $\wedge E$
α_4	(8)	$\neg(\neg p \wedge \neg q)$	4,7[α_1] RAA
α_2	(9)	$\neg(\neg p \wedge \neg q)$	2,6[α_3],8[α_4] $\vee E$
α_1	(10)	$\neg(p \vee q)$	1,9[α_2] RAA
α_1	(n)	$\neg(p \vee q)$	x,y[α_2] RAA

$X \vdash A$, A is an Atom (Or: all Other Strategies Fail)

If everything else fails, assume $\neg A$ and derive A: $\dfrac{X, \neg A \vdash A}{X \vdash A}$

Closely related is the sequent from earlier: $\neg p \rightarrow p \vdash p$
(p is so true that it's even implied by its own negation!)

Example: $(p \rightarrow q) \rightarrow p \vdash p$

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B} RAA$$

α_1	(1)	$(p \rightarrow q) \rightarrow p$	A
α_2	(2)	$\neg p$	A
α_3	(3)	p	A
α_2, α_3	(4)	$\neg\neg q$	2,3[] RAA
α_2, α_3	(5)	q	4 $\neg\neg E$
α_2	(6)	$p \rightarrow q$	5[α_3] $\rightarrow I$
α_1, α_2	(7)	p	1,6 $\rightarrow E$
α_1	(8)	$\neg\neg p$	2,7[α_2] RAA
α_1	(9)	p	8 $\neg\neg E$

Examples for Semantic Tableau

Previous Example, shown with Semantic Tableau

We now show $(p \to q) \to p \vdash p$ via Semantic Tableau.

T: $A \to B$
F: A | **T:** B

F: $A \to B$
T: A , **F:** B

$(p \to q) \to p \vdash p$

(1) **T:** $(p \to q) \to p$ ✓
(2) **F:** p

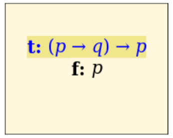
(3) **F:** $p \to q$ ✓ from (1)
(5) **T:** p ↯ from (3)
(6) **F:** q from (3)

(4) **T:** p ↯ from (1)

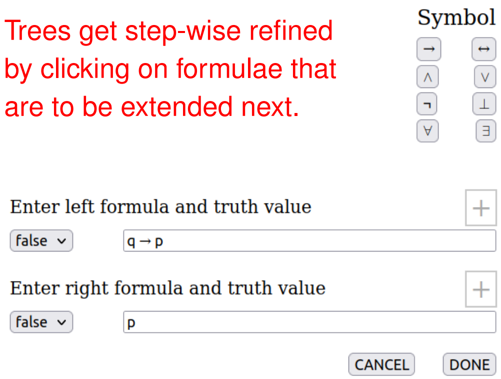
The primary strategy (that often suffices to create small trees) is:

- *Always* apply rules first that don't branch!
- Don't forget branches!! And mark lines that are "done".

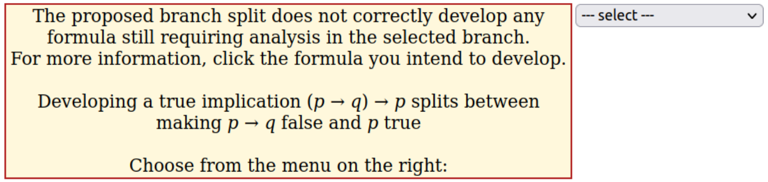
Tool Support (Illustrated on Previous Example)



Trees get step-wise refined by clicking on formulae that are to be extended next.

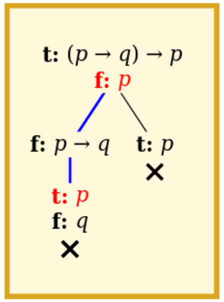


The marked formula was expanded in a wrong way here!



Tool Support (Illustrated on Previous Example), cont'd

The final, completed tree looks as follows:



Finished: all branches closed

- Like the ND proof checker, this one will also detect all problems (errors).
- The tree is constructed step by step by clicking on non-processed formulae.
- By clicking underneath a branch, the tool states whether it's open or closed.
- The link to the checker (and its manual) can also be found on Wattle.

Summary

Content of this Lecture

- Today, we did a recap on how to prove various kinds of sequents via Natural Deduction and Semantic Tableau
- We also (briefly) introduced some tool support:
  - Proof checker for Natural Deduction proofs
  - Proof checker for Semantic Tableau proofs