Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: First-Order Logic
— Introduction and Natural Deduction

Pascal Bercher

AI Group
School of Computing
College of Engineering and Computer Science
the Australian National University

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Australian National University

Introduction

Motivation

How to model that in propositional logic?

- All logicians are rational — aLr
- Some philosophers are not rational — ¬ sPr

} premises

- Thus, not all philosophers are logicians — $\neg aPL$

} conclusion

So, can we prove $aLr, \neg sPr \vdash \neg aPL$?

- No! It's even three completely different propositions!
- We need a more expressive logic!

(We could have also used p, q, and r above, the names above were chosen to have more "speaking" names.)

How to extend Propositional Logic?

Logic is about making statements: Socrates is a goat

(natural language) sentence: "Socrates is a goat"; constant: "Socrates"; predicate: "is a goat"

In first-order logic, we:

- can represent individual objects (people, goats, footballs, etc.)
- and express properties and relationships between objects.

In our example,

- the "object" *Socrates* can be represented by a constant,
- the "property" *is a Goat* can be represented by a predicate.

⇒ For example, *isGoat*(*Socrates*)

⇒ In propositional logic, we had to use *SocratesIsGoat*, which is missing some information, since it does not "relate" to another proposition involving Socrates, like *SocratesKicksGoat*. (Also cf. previous example with philosophers and logicians! Same issue!)

Predicate Logic

Terminology And Conventions: Terminology

Term: Anything that represents an object, i.e.,

- a constant (representing a fixed object, like the person Socrates)
- a variable (representing a non-specified object)
- a function (representing a fixed object given a sequence of terms)

Intuition:

- Constants are meant to represent concrete objects, as in "*isGoat*(*Socrates*)".
- Variables are used for "more general" relationships as in: "All logicians are rational". They are basically placeholders.

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- a function (representing a fixed object given a sequence of terms)

Predicates: Express properties or relations of/between terms:

- Takes as input (or "argument") a sequence of terms.
  - The sequence length depends on the predicate, e.g., *isGoat* is unary, *kicks* is binary, etc. (some might even be nullary!)
  - This length is called *arity* and can be given as a subscript, e.g., $isGoat_1$, $kicks_2$, but we don't since it's clear from context.
- Maps to a truth value, e.g., *isGoat*(*Socrates*) might be false, but *isGoat*(*Jimmy*) might be true.

⇒ The "formal semantics" (e.g., for which terms is a predicate true?) will be given in week 7.

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Terminology And Conventions: Conventions

- We continue to use our sequent notation!
  - $X \vdash A$
  - $X, Y \vdash A$
  - $X, A \vdash B$
  - etc. Only now they represent first-order predicate logic formulae.
- As before we write only single letters!
  - X, Y, Z for *sets* of formulae, and
  - A, B, C for *single* formulae.

Terminology And Conventions: Conventions (cont'd)

- Capital letters are predicate symbols:
  $F, G, H, \ldots, P, Q, R, L, \ldots$
- Lower-case letters represent terms:
  - a, b, c are (usually) used for constants, but we also use them for free variables (as they behave in the same way).
  - f, g, h are used for functions.
  - v and x, y are used for variables.
  - t is used for terms (i.e., any of the above).
- For the sake of simplicity, we do not use parentheses, e.g., $F(a)$, $G(b)$, and $R(a, b)$ become Fa, Gb, and Rab, respectively.
- Now it's clear that the arity is clear from the context! E.g.,
  - Fa represents a predicate F with arity 1 (with term a), and
  - Rab represents a predicate R with arity 2 (with terms a and b).

First-Order Formulae: Introduction

Example:

- All vulcans are logicians.
- Let's reformulate that step by step:
  - Take any vulcan, call it x, then x is a logician. (x is a variable!)
  - Take any vulcan, call it x, then Lx.
  - For every object x, such that x is a Vulcan, Lx holds.
  - For every object x, such that Vx holds, Lx holds.
- Now we need special syntax for that "every object"!

First-Order Formulae: Possible Quantifiers

We want to "quantify" the objects we talk about.

- For every object x, such that Vx holds, Lx holds.
- *More formally:* $\underbrace{ALL(x : Vx)}_{\text{quantifier!}} Lx$
- *Even more formally:* $\underbrace{\forall}_{\text{quantity indicator}} (\underbrace{x}_{\text{variable}} : \underbrace{Vx}_{\text{sort indicator}}) Lx$

What other quantifiers could possibly exist?

- $SOME(x : Vx)Ex$ ("some vulcans are emotional")
- $FEW(x : Vx)Ex$ ("a few vulcans are emotional")
- $MOST(x : Lx)\neg Vx$ ("most logicians are not vulcans")
- $ONE(x : Lx)(Ex \land Vx)$ ("One logician is an emotional vulcan")

First-Order Formulae: Possible Quantifiers

We want to "quantify" the objects we talk about.

- For every object x, such that Vx holds, Lx holds.
- *More formally:* $\underbrace{ALL(x : Vx)}_{\text{quantifier!}}\, Lx$
- *Even more formally:* $\underbrace{\forall}_{\text{quantity indicator}} (\underbrace{x}_{\text{variable}} : \underbrace{Vx}_{\text{sort indicator}}) Lx$

What quantifiers *do* exist? (In our predicate logic!)

- Just two!
- $ALL(x : A)B$, i.e., $\forall(x : A)B$
- $SOME(x : A)B$, i.e., $\exists(x : A)B$

"SOME" means "at least one", so "$\exists$" is also called "exists"
"ALL", i.e., $\forall$, is called the "universal" quantifier

First-Order Formulae: Example (from before)

Propositional logic (not working):

- All logicians are rational aLr
- Some philosophers are not rational $\neg sPr$
- Thus, not all philosophers are logicians $\neg aPL$

Predicate logic (works!):

- All logicians are rational $\forall(x : Lx)Rx$
- Some philosophers are not rational $\exists(x : Px)\neg Rx$
- Thus, not all philosophers are logicians $\neg\forall(x : Px)Lx$

But how to prove "$\forall(x : Lx)Rx, \exists(x : Px)\neg Rx \vdash \neg\forall(x : Px)Lx$"?

- Natural Deduction
- Semantic Tableau

Examples: Some Small Examples

- All goats are hairy. $\forall(x : Gx)Hx$
- Some footballers are hairy. $\exists(x : Fx)Hx$
- No goats are footballers. $\neg\exists(x : Gx)Fx \equiv \forall(x : Gx)\neg Fx$

You can see here that negations before formulae invert the outer-most quantifier an get moved before the inner formula (which might again be a quantified formula).

You can prove this "rule", but you can't use it!

Examples: More Complicated Examples

- Every hairy footballer kicks a goat.
  - $\forall(x : Hx \land Fx)\exists(y : Gy)Kxy$
  - $\exists(x : Gx)\forall(y : Hy \land Fy)Kyx$
  - Are these the same?
  - In the first formula, each footballer may kick his/her own goat!
    In the second, all footballers kick the same goat!*
    * I claim that this model is *wrong*! This is not what the sentence is saying; the second formula is more/too specific.
- Only hairy footballers kick goats.
  - $\forall(x : Fx \land \exists(y : Gy)Kxy)Hx$
  - $\forall(x : \exists(y : Gy)Kxy)(Hx \land Fx)$
  - The first model means: "All footballers that kick a goat are hairy."
    The second: "Anything that kicks a goat is a hairy footballer."
    But which one is right? (Hard to tell, language is vague!)

Examples: Note on Interpreting Natural Language

- Modeling a proposition with logic often reveals how vague language is! (And that we may make implicit assumptions)
- "Only hairy footballers kick goats." What is meant here?
  - Among all footballers, only the hairy ones kick goats.
  - Among all human beings, only hairy footballers kick goats.
- "Only unmotivated students don't study." What is meant here?
  - Among all *students*, only the unmotivated ones don't study.
  - Among all *human beings* (or all aliens, spirits, . . . ?), only unmotivated students don't study.
- We normally rely on context to figure out what's meant. But when you are (t)asked to formalize something you need to be as formal as possible.

Introduction Predicate Logic Natural Deduction Universal Quantifiers Existential Quantifiers Summary

Examples: Translating a Natural Language Text into Predicate Logics

Anyone who sees a hairy footballer sees someone who kicks a non-footballer.

Sounds horribly complicated, but let's do it step by step!

1. $\forall(x : x$ sees a hairy footballer$)$
   $(x$ sees someone who kicks a non-footballer$)$
2. $\forall(x : \exists(y : Hy \land Fy)Sxy)$
   $(\quad \exists(y : y$ kicks a non-footballer$)Sxy \quad)$
3. $\forall(x : \exists(y : Hy \land Fy)Sxy)$
   $(\quad \exists(y : \quad \exists(z : \neg Fz)Kyz \quad)Sxy \quad)$

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Examples: Translation: More examples! (From You)

Novel/recent idea:

- We create a new forum for the formalization of Natural Language!
- We can use it to create some examples to practice this!
- One thread will be one Natural Language
  – plus the attempt(s) to formalize it!
- Feedback can be provided by other students;
  and maybe occasionally by Lecturers
- The idea is basically an asynchronious online learning group
- And some of the completed/correct (most funny?) examples could be preserved for next generations!
- (I'm thinking of additional bonus material just like the many practice sequents in the Logic Notes; see the link in the bonus material sequence.)

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Well-formed Formulae: Restrictions on Formulae, cont'd

Recap of Propositional Logic:

- Wellformed formula: $(p \lor q) \to (p \land q)$
  (though it's not a tautology/theorem, but that's not the point)
- Non-wellformed formula: $(p \lor \to q) \land q\neg)$

So what about Predicate Logic?

- Connectives and subformulae are used in the same way as for Propositional Logic.
- Some additional restrictions (next slide)

Well-formed Formulae: Restrictions on Formulae

Predicate Logic:

- Example for two well-formed formulae:
  - $\forall(x : Fx)\exists(y : Gy)Kxy$ — Each footballer kicks a goat
  - $\exists(x : Fx)\exists(y : Gy)Kxy$ — Some footballer kicks a goat
- What about $(\forall(x : Fx)\exists(y : Gy)Kxy) \wedge (\exists(x : Fx)\exists(y : Gy)Kxy)$?
  $\rightarrow$ Still allowed! After all, you can put them together with $\wedge I$!
- So, are there *any* restrictions?
  - Let A and B (well-formed) formulae and x free in A or B, then
    - $\forall(x : A)B$ and $\exists(x : A)B$ are (well-formed) formulae.
  - I.e., we don't allow quantification over non-used variables!
  - (We don't provide a complete specification of what's well-formed, just as we didn't for propositional logic.)

From Restricted Quantifiers to Unrestricted Quantifiers: Main Idea

So far, we were only considering *restricted quantifiers*:

- $\exists(x : Px)\neg Rx$ (Some philosophers are not rational)
- $\forall(x : Lx)Rx$ (All logicians are rational)

But now – to make life easier! – we don't make restrictions anymore!

We cheat to get around having a sort restriction:
$\exists(x : x \text{ is a thing})Gx$ is the same as $\exists(x : Fx \rightarrow Fx)Gx$

Example:

- Something is hairy:
  - $\exists(x : Fx \rightarrow Fx)Hx$
  - So we could also just write $\exists x\ Hx$
- Some goats are hairy:
  - $\exists(x : Gx)Hx$
  - So we can just write $\exists x\ Gx \wedge Hx$

From Restricted Quantifiers to Unrestricted Quantifiers: Eliminating Sort Indicator

Did we lose something when switching to unrestricted quantifiers?

No! (So, we can just use unrestricted logic instead!)

- Existential quantified formulae become conjunctions:
  E.g., $\exists(x : Gx)Hx$ (some goats are hairy) becomes $\exists x\ Gx \wedge Hx$
- Universally quantified formulae become implications:
  E.g., $\forall(x : Gx)Hx$ (all goats are hairy) becomes $\forall x\ Gx \rightarrow Hx$

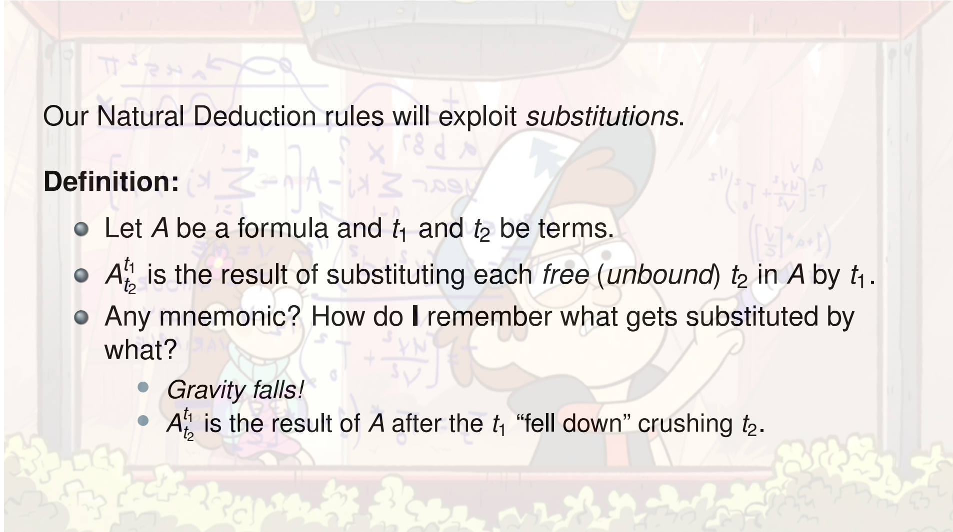
Thus, from now on, we will use non-restricted formulae instead.
(But in the second half of the course we will re-visit restricted quantifiers again.)

Natural Deduction

Introduction

- Instead of re-doing all our previous rules, we will just provide *additional ones*!
  - Two new rules for $\forall$ (introduction and elimination)
  - Two new rules for $\exists$ (introduction and elimination)
- We still perform natural deduction for *propositional logic* in intermediate steps.

Substitutions: Introduction



Our Natural Deduction rules will exploit *substitutions*.

Definition:

- Let A be a formula and t_1 and t_2 be terms.
- $A^{t_1}_{t_2}$ is the result of substituting each *free* (*unbound*) t_2 in A by t_1.
- Any mnemonic? How do I remember what gets substituted by what?
  - *Gravity falls!*
  - $A^{t_1}_{t_2}$ is the result of A after the t_1 “fell down” crushing t_2.

Substitutions: Examples (and Conventions)

- Let $A = \exists x(Px \to Rx)$. Is $A^y_x = \exists y(Py \to Ry)$?
  - No! Recall that x is required to be free/unbound in A!
  - Since there are no free variables, so $A^y_x = A$ here.
- Let $A = Fx \land \exists x(Fx \land Gx)$. What's A^y_x now?
  - It's $Fy \land \exists x(Fx \land Gx)$!
  - Because we only substitute free/unbound variables!
- You won't have to take much care as we will use a convention to use the constant letters a, b, c for free variables in all our proofs.
- E.g., you might see something like $A = Fa \land \exists x(Fx \land Gx)$, but *never* something like $Fx \land \exists x(Fx \land Gx)$ or even $\exists a(Fa \land Ga)$.
- We use this because constants “behave” just like free variables. In fact, the Logic notes never even use free variables! It only uses bound variables and constants (called *names* there).

Universal Quantifiers

Universal Elimination: Introduction

- Let's assume we want to say that the age of all humans is smaller than 130: $\forall x\ age(x) < 130$
- If we had one constant for each individual (person), we could conclude: $age(a) < 130 \wedge age(b) < 130 \wedge age(c) < 130 \wedge \ldots$ (Though that's clearly not practical! And maybe not even possible if we reason about infinitely many objects like numbers.)
- So we could also conclude $age(x) < 130$ for *any* x! We thus use a (free) *variable* in our rule!
- So, what will the Universal-Elimination rule look like?

$$\frac{\forall x\ Fx}{Fv}\forall E \qquad \text{more general:} \qquad \frac{\forall x\ A}{A^t_x}\forall E$$

We do however need a side condition here to make sure our newly introduced term doesn't cause trouble.

Universal Elimination: Side Condition

Assume we had no side condition:

$$\frac{\forall x\ A}{A^t_x}\forall E \qquad \text{in sequent notation:} \qquad \frac{X \vdash \forall x\ A}{X \vdash A^t_x}\forall E$$

Let's consider this sequent: $\forall x\ \exists y(y > x) \vdash \exists y(y > y)$

- Should that be valid? No! No number is larger than itself!
- But we can prove it! (If there's no side condition!)

α_1	(1)	$\forall x\ \exists y(y > x)$	A
α_1	(2)	$\exists y(y > y)$	1 $\forall E$

$$\frac{\forall x\ \overbrace{\exists y\ (y > x)}^{A}}{\underbrace{\exists y\ (y > y)}_{A^t_x \equiv A^y_x}}\forall E$$

So what's missing?

- The "instantiation of x" (the new variable name) must be free! (We don't want it to get captured by another quantifier!)
- This is different from what we demanded for substitutions.

Universal Elimination: The 1-step Rule

So, in conclusion:

Universal Elimination Rule:

$$\frac{X \vdash \forall x\ A}{X \vdash A^t_x}\forall E \quad \text{only if } t \text{ is not bound in } A^t_x!$$

As mentioned earlier (slide 23), you are not in risk of making that mistake as long as you adhere our convention: use a, b, c, for free variables!

Important note:
Recall that often you *apply* the rule from bottom to top!

- E.g., you might have some line X (i) $\exists y(y > a)$, and then
- you apply $\forall E$ to (i) to obtain: X (i-1) $\forall x \exists y(y > x)$!

Universal Introduction: Introduction

- For the *introduction* of the universal quantifier, we would like to have, conceptually, a rule like the following:

$$\frac{Fa \quad Fb \quad Fc \quad \ldots}{\forall x\ Fx}\forall I$$

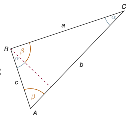
But that's again infeasible, and potentially even impossible!

How about: $\frac{Fa}{\forall x\ Fx}\forall I$? (as above, a is a constant)

That rule is wrong! Just because Aristotle is (was) a footballer, doesn't mean that everybody is!

But it might work for "typical objects"... (a variable)

Universal Introduction: Typical Objects

- What's a typical object? (A free variable)
- Remember the "undergraduate school" when you have to proof some property of *all* triangles.
  - *Step 1:* Let ABC=  be a triangle.
  - *Step 2:* "some fancy proof"
  - *Step 3:* Thus, ABC has property P. Thus P holds for all triangles!
- Why is that correct? Since we did not make any assumptions for ABC other than it being a triangle! E.g., we did not demand that it has a 90-degree angle or any other special case! We gave it a name (ABC), but that was also arbitrary!

Universal Introduction: The 1-step Rule

- So, we need an "object without any assumption" to generalize its property (formula) to the general case.
- But how to express this "no assumptions"?
- $\dfrac{Fv}{\forall x\ Fx}\forall I$ more general: $\dfrac{A}{\forall x\ A^x_v}\forall I$ with side condition: provided the variable v does not occur in any assumption that A depends upon.
- **Universal Introduction Rule**: (in sequent notation)

$$\frac{X \vdash A}{X \vdash \forall x\ A^x_v}\forall I \quad \text{only if } v \text{ does not occur in } X!$$

Universal Introduction: More on our Assumption and Side-Conditions

So we have: $\dfrac{X \vdash A}{X \vdash \forall x\ A^x_v}\forall I$ only if v does not occur in X!

So, can we use $\dfrac{Fv}{\forall x\ Fx}\forall I$ to prove $Faristotle \vdash \forall x\ Fx$?

Let's try! Does not work: Rule fails for two reasons!

$Faristotle \vdash \forall x\ Fx$

α_1	(1)	$Faristotle$	A
α_1	(n)	$\forall x\ Fx$	

$\dfrac{X \vdash A}{X \vdash \forall x\ A^x_v}\forall I$

Only if v does not occur in X!

Universal Introduction: More on our Assumption and Side-Conditions

So we have: $\dfrac{X \vdash A}{X \vdash \forall x\ A^x_v}\forall I$ only if v does not occur in X!

So, can we use $\dfrac{Fv}{\forall x\ Fx}\forall I$ to prove $Faristotle \vdash \forall x\ Fx$?

- Two reasons this "proof" (luckily) fails:
  - The side condition states that $v = aristotle$ (which is gets substituted) does not occur in X, but X is $\alpha_1 = Faristotle$, so the side condition is violated.
  - Also, in our rule above v represents a variable. So it's not applicable here anyway. Recall: Just because Aristotle plays football, not everybody does!

More on Syntax: Meaning of $a, b, c, \ldots$ in Proofs

Important:

- Note that the Logic Notes (and Yoshi as well!) refer to constants as "names".
- More importantly, note that we only use a, b, c, etc. as constants in the explanations for rules here in the lecture. For convenience, in all *exercises*, including those given in the lecture, these letters do *not* represent constants!
- Instead, in formal proofs, these letters (again: $a, b, c, \ldots$) will represent our "typical objects" – like the triangle ABC from before!
- These "typical objects" are (free, i.e., unbound) variables.
- In fact, you will never deal with constants in proofs!

Example Proofs: Example 1

$$\forall x\, Fx, \forall x\, Gx \vdash \forall x\, (Fx \wedge Gx)$$

α_1	(1)	$\forall x\, Fx$	A
α_2	(2)	$\forall x\, Gx$	A
α_1	(3)	Fa	1 $\forall E$
α_2	(4)	Ga	2 $\forall E$
α_1, α_2	(5)	$Fa \wedge Ga$	3,4 $\wedge I$
~~α_1, α_2~~	~~(6)~~	~~$\forall x\, (Fx \wedge Gx)$~~	~~5 $\forall I$~~
α_1, α_2	(n-1)	$Fa \wedge Ga$	
α_1, α_2	(n)	$\forall x\, (Fx \wedge Gx)$	(n-1) $\forall I$

$$\frac{X \vdash A}{X \vdash \forall x\, A^x_v}\forall I$$

Only if v does not occur in X!

$$\frac{X \vdash \forall x\, A}{X \vdash A^t_x}\forall E$$

Example Proofs: Example 1

$$\forall x\, Fx, \forall x\, Gx \vdash \forall x\, (Fx \wedge Gx)$$

α_1	(1)	$\forall x\, Fx$	A
α_2	(2)	$\forall x\, Gx$	A
α_1	(3)	Fa	1 $\forall E$
α_2	(4)	Ga	2 $\forall E$
α_1, α_2	(5)	$Fa \wedge Ga$	3,4 $\wedge I$
α_1, α_2	(6)	$\forall x\, (Fx \wedge Gx)$	5 $\forall I$

$$\frac{X \vdash A}{X \vdash \forall x\, A^x_v}\forall I$$

Only if v does not occur in X!

$$\frac{X \vdash \forall x\, A}{X \vdash A^t_x}\forall E$$

Did we adhere all side conditions? Yes!

- $X \vdash A$ of the $\forall I$ rule corresponds to line 5, which is $\alpha_1, \alpha_2 \vdash Fa \wedge Ga$,
- variable v corresponds to a, and
- although a (of course!) occurs in $Fa \wedge Ga$, it is not in $X = \{\alpha_1, \alpha_2\} = \{\forall x\, Fx, \forall x\, Gx\}$, so all good!

Example Proofs: Example 2

$$\forall x\, (Fx \rightarrow \forall y\, Fy) \vdash \forall x\, (\neg Fx \rightarrow \forall y\, \neg Fy)$$

α_1	(1)	$\forall x\, (Fx \rightarrow \forall y\, Fy)$	A
α_2	(2)	$\neg Fa$	A

Why did we substitute y by b rather than by a?

- $X \vdash A$ corresponds to $\neg Fb$,
- The variable v in the rule (which is b in our case!) may not occur in X, which works for us since $X = \{\alpha_1, \alpha_2\} = \{\forall x\, (Fx \rightarrow \forall y\, Fy), \neg Fa\}$
- So choosing $v = a$ would not have been possible, since a occurs in $\alpha_2 = \neg Fa$!

$$\frac{X \vdash A}{X \vdash \forall x\, A^x_v}\forall I$$

Only if v does not occur in X!

α_1, α_2	(n-3)	$\neg Fb$	
α_1, α_2	(n-2)	$\forall y\, \neg Fy$	(n-3) $\forall I$
α_1	(n-1)	$\neg Fa \rightarrow \forall y\, \neg Fy$	(n-2)[α_2] $\rightarrow I$
α_1	(n)	$\forall x\, (\neg Fx \rightarrow \forall y\, \neg Fy)$	(n-1) $\forall I$

Example Proofs: Example 2

$\forall x\,(Fx \to \forall y\, Fy) \vdash \forall x\,(\neg Fx \to \forall y\, \neg Fy)$

α_1	(1)	$\forall x\,(Fx \to \forall y\, Fy)$	A
α_2	(2)	$\neg Fa$	A
α_3	(3)	Fb	A
α_1	(4)	$Fb \to \forall y\, Fy$	1 $\forall E$
α_1, α_3	(5)	$\forall y\, Fy$	3,4 $\to E$
α_1, α_3	(6)	Fa	5 $\forall E$
α_1, α_2	(7)	$\neg Fb$	2,6[α_3] RAA
α_1, α_2	(8)	$\forall y\, \neg Fy$	7 $\forall I$
α_1	(9)	$\neg Fa \to \forall y\, \neg Fy$	8[α_2] $\to I$
α_1	(10)	$\forall x\,(\neg Fx \to \forall y\, \neg Fy)$	9 $\forall I$

α_1, α_2	(n-3)	$\neg Fb$	x,y[α_3] RAA
α_1, α_2	(n-2)	$\forall y\, \neg Fy$	(n-3) $\forall I$
α_1	(n-1)	$\neg Fa \to \forall y\, \neg Fy$	(n-2)[α_2] $\to I$
α_1	(n)	$\forall x\,(\neg Fx \to \forall y\, \neg Fy)$	(n-1) $\forall I$

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}\,RAA$$

$$\frac{X \vdash A}{X \vdash \forall x\, A^x_v}\,\forall I$$

Only if v does not occur in X!

$$\frac{X \vdash \forall x\, A}{X \vdash A^t_x}\,\forall E$$

Existential Quantifiers

Existential Introduction: Introduction

- Recall that you can “imagine” the universal quantifier $\forall$ like: $age(a) < 130 \wedge age(b) < 130 \wedge age(c) < 130 \wedge \ldots$
- The existential quantifier $\exists$ can similarly interpreted as: $age(a) > 100 \vee age(b) > 100 \vee age(c) > 100 \vee \ldots$
- Thus, conceptually, we would expect something like the following rule: (a, b, ... are again constants)

$$\frac{Fa \vee Fb \vee Fc \vee \ldots}{\exists x\, Fx}\,\exists I$$

Existential Introduction: The 1-step Rule (part I)

- **Existential Introduction Rule:**

$$\frac{Fv}{\exists x\, Fx}\,\exists I \quad \text{more general:} \quad \frac{A^t_x}{\exists x\, A}\,\exists I \quad \text{in sequent notation:} \quad \frac{X \vdash A^t_x}{X \vdash \exists x\, A}\,\exists I$$

- This rule assumes a non-empty “universe” (the objects that we reason about, more later when we formally deal with the semantics), i.e., that there exists at least one “object” that the terms represent. This is one of several assumptions in ***classical logic***, though there are other important properties as well.
- Just like $\forall E$, this rule also has a side condition! Let’s see in an example which and why.

Existential Introduction: Side Condition

Assume we had no side condition:

$$\frac{A^t_x}{\exists x\ A}\exists I \qquad \text{in sequent notation:} \qquad \frac{X \vdash A^t_x}{X \vdash \exists x\ A}\exists I$$

Let's consider this sequent: $\forall y\ (y = y) \vdash \exists x \forall y\ (y = x)$

- Should that be valid? No! There is not just one number! :)
- But we can prove it! (If there's no side condition!)

α_1	(1)	$\forall y\ (y = y)$	A
α_1	(2)	$\exists x \forall y (y = x)$	1 $\exists I$

$$\frac{\overbrace{\forall y\ (y = y)}^{A^t_x \equiv A^y_x}}{\exists x\ \underbrace{\forall y\ (y = x)}_{A}}\exists I$$

So what's the problem?

- We were quantifying an already bound variable! (The right y.)
- We were missing: The x in A must be free (as y) in A^y_x.
- *Not any issue at all as long as you follow our convention!*

Existential Introduction: The 1-step Rule (part II)

So, in conclusion:

Existential Introduction Rule:

$$\frac{X \vdash A^t_x}{X \vdash \exists x\ A}\exists I \qquad \text{only if } t \text{ is not bound in } A^t_x$$

As mentioned earlier (slide 23), you are not in risk of making that wrong as long as you adhere our convention: use a, b, c, for free variables!

Important note:

Recall that often you *apply* the rule from bottom to top!

- E.g., you might have some line X (i) $\exists y(Fa \rightarrow Fy)$, and then
- you apply $\exists I$ to (i) to obtain: X (i-1) $Fa \rightarrow Fa$!

Existential Elimination: Introduction

- We want to *eliminate* the existential quantifier. So can we just use the following rule? $\dfrac{\exists x\ Fx}{Fv}\exists E$? Recall: $\dfrac{\forall x\ Fx}{Fv}\forall E$!
- So, no! Because we don't know *which* object has that property! (You can try to "prove" some invalid sequent when having this (wrong) rule available!)

Existential Elimination: Introduction, cont'd

- The *idea* behind the rule is the following:

$$\frac{\exists x\ Fx \qquad \begin{matrix}[Fy]\\ \vdots \\ B\end{matrix}}{B}\exists E \qquad \text{for a "typical" } y.$$

- The idea is similar to disjunction elimination: In $A \vee B$, we don't know whether A or B is true, so we assume both and show that either way the derivation can be done.
- Here, we show it for "some instance" that does not pose further restrictions (and then discharge it since we know that such an "instance" exists due to the assumption $\exists x\ Fx$).

Existential Elimination: The 1-step Rule

Existential Elimination Rule:

$$\frac{X \vdash \exists x\, A^x_t \qquad Y, A \vdash B}{X, Y \vdash B}\exists E$$

Provided t does not occur in B or any formula in Y.

- Note what's written here: The assumption formula A in sequent 2 can be regarded an "instantiation" of the derivation in sequent 1 by substituting x by a term.
- We need the side condition so that our choice of the "instance" of x is still "general".
- Otherwise we might be able to derive simply because we chose a specific special case!
- Again, you can try to prove an invalid sequent, which you might be able to if you violate that side condition!

Examples: Example 1

$\vdash \forall x\, \exists y\, (Fx \rightarrow Fy)$

α_1	(1)	Fa	A
	(2)	$Fa \rightarrow Fa$	1[α_1] $\rightarrow I$
	(3)	$\exists y\, (Fa \rightarrow Fy)$	2 $\exists I$
	(n-1)	$\exists y\, (Fa \rightarrow Fy)$	
	(n)	$\forall x\, \exists y\, (Fx \rightarrow Fy)$	(n-1) $\forall I$

$$\frac{X \vdash A}{X \vdash \forall x\, A^x_v}\forall I$$

Only if v does not occur in X!

$$\frac{X \vdash A^t_x}{X \vdash \exists x\, A}\exists I$$

- Wait a minute! Didn't we say that A^t_x replaces *all* occurrences of x in A by t?

  So, going from line (2) $Fa \rightarrow Fa$ to line (3) $\exists y\, (Fa \rightarrow Fy)$ is wrong, right?

$$\frac{\overbrace{Fa \rightarrow Fa}^{A^t_x \equiv A^a_y}}{\exists y\; \underbrace{(Fa \rightarrow Fy)}_{A}}\exists I$$

- No! We *did* replace all x (here: y) by t (here: a)! (See illustration.)

Examples: Example 1

$\vdash \forall x\, \exists y\, (Fx \rightarrow Fy)$

α_1	(1)	Fa	A
	(2)	$Fa \rightarrow Fa$	1[α_1] $\rightarrow I$
	(3)	$\exists y\, (Fa \rightarrow Fy)$	2 $\exists I$
	~~(4)~~	~~$\forall x\, \exists y\, (Fx \rightarrow Fy)$~~	~~3 $\forall I$~~
	(n-1)	$\exists y\, (Fa \rightarrow Fy)$	
	(n)	$\forall x\, \exists y\, (Fx \rightarrow Fy)$	(n-1) $\forall I$

$$\frac{X \vdash A}{X \vdash \forall x\, A^x_v}\forall I$$

Only if v does not occur in X!

$$\frac{X \vdash A^t_x}{X \vdash \exists x\, A}\exists I$$

Examples: Example 2

$\exists x\, (Fx \wedge Gx) \vdash \exists x\, Fx \wedge \exists x\, Gx$

α_1	(1)	$\exists x\, (Fx \wedge Gx)$	A
α_2	(2)	$Fa \wedge Ga$	A
α_2	(3)	Fa	2 $\wedge E$
α_2	(4)	$\exists x\, Fx$	3 $\exists I$
α_2	(5)	Ga	2 $\wedge E$
α_2	(6)	$\exists x\, Gx$	5 $\exists I$
α_2	(7)	$\exists x\, Fx \wedge \exists x\, Gx$	4,6 $\wedge I$
~~α_1~~	~~(8)~~	~~$\exists x\, Fx \wedge \exists x\, Gx$~~	~~1,7[α_2] $\exists E$~~
α_2	(n-1)	$\exists x\, Fx \wedge \exists x\, Gx$	
α_1	(n)	$\exists x\, Fx \wedge \exists x\, Gx$	1,(n-1)[α_2] $\exists E$

$$\frac{X \vdash \exists x\, A^x_t \qquad Y, A \vdash B}{X, Y \vdash B}\exists E$$

Provided t does not occur in B or any formula in Y.

$$\frac{X \vdash A^t_x}{X \vdash \exists x\, A}\exists I$$

Examples: Example 3: On Logicians and Philosophers

Remember from the beginning:

- All logicians are rational $\forall(x : Lx)Rx$
- Some philosophers are not rational $\exists(x : Px)\neg Rx$
- Thus, not all philosophers are logicians $\neg\forall(x : Px)Lx$

Now, in our unsorted predicate logic, this is:

- All logicians are rational $\forall x\ Lx \to Rx$
- Some philosophers are not rational $\exists x\ Px \wedge \neg Rx$
- Thus, not all philosophers are logicians $\neg\forall x\ Px \to Lx$

Examples: Example 3: On Logicians and Philosophers (cont'd)

$\forall x\ Lx \to Rx, \exists x\ Px \wedge \neg Rx \vdash \neg\forall x\ Px \to Lx$

α_1	(1)	$\forall x\ Lx \to Rx$	A
α_2	(2)	$\exists x\ Px \wedge \neg Rx$	A
α_3	(3)	$Pa \wedge \neg Ra$	A
α_4	(4)	$\forall x\ Px \to Lx$	A
α_4	(5)	$Pa \to La$	4 $\forall E$
α_3	(6)	Pa	3 $\wedge E$
α_3, α_4	(7)	La	5,6 $\to E$
α_1	(8)	$La \to Ra$	1 $\forall E$
$\alpha_1, \alpha_3, \alpha_4$	(9)	Ra	7,8 $\to E$
α_3	(10)	$\neg Ra$	3 $\wedge E$
α_1, α_3	(11)	$\neg\forall x\ Px \to Lx$	9,10[α_4] RAA
~~α_1, α_2~~	~~(12)~~	~~$\neg\forall x\ Px \to Lx$~~	~~2,11[α_3] $\exists E$~~
α_1, α_3	(n-1)	$\neg\forall x(Px \to Lx)$	x,y[α_4] RAA
α_1, α_2	(n)	$\neg\forall x(Px \to Lx)$	2,(n-1)[α_3] $\exists E$

$$\frac{X \vdash \exists x\ A^x_t \qquad Y, A \vdash B}{X, Y \vdash B}\exists E$$

Provided t does not occur in B or any formula in Y.

$$\frac{X, B \vdash A \qquad Y, B \vdash \neg A}{X, Y \vdash \neg B}RAA$$

Summary

Content of this Lecture

- We introduced predicate logic:
  - with restricted quantifiers (we re-visit this later)
  - and with unrestricted quantifiers (default!)
- Predicate logic can reason about objects!
- Natural deduction for predicate logics, with additional rules for:
  - Introduction and Elimination rules for $\forall$ and $\exists$
  - For the rest we keep using the rules from propositional logics!
- Many side conditions...
  - Substitutions: only rename free variables
  - $\forall E$ and $\exists I$: Just make sure to follow naming conventions :)
  - $\forall I$ and $\exists E$: They are more complicated, look them up!

→ The entire Logic Notes sections:
  - 4: Expressing Generality
    - except “Properties of relations”
    - and except “Functions”
    - (You should read them anyway, in particular “Functions”!)