

Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: First-Order Logic
— Properties of Proof Systems and Semantic Tableaux

Pascal Bercher

AI Group
School of Computing
College of Engineering and Computer Science
the Australian National University

pre-recorded for: 28 & 29 March 2022



Australian
National
University

Introduction

Recap: Predicate Logics

- We now (since week 5) know Predicate Logics as a means to express properties of and relationships between objects.
- For example:
 - If everyone plays football, and everyone is a goat, then everyone is a football-playing goat
 - $\forall x Fx, \forall x Gx \vdash \forall x (Fx \wedge Gx)$

Recap: Predicate Logics

- We now (since week 5) know Predicate Logics as a means to express properties of and relationships between objects.
- For example:
 - If everyone plays football, and everyone is a goat, then everyone is a football-playing goat
 - $\forall x Fx, \forall x Gx \vdash \forall x (Fx \wedge Gx)$
- We know how to prove sequents involving Predicate Logic using Natural Deduction.
 - We “only” needed additional elimination and introduction rules for the exists (\exists) and universal (\forall) quantifiers.
 - Other than that we just re-used the rules for Propositional Logic.

Recap: Semantic Tableau

- Today, we cover Semantic Tableau for *Predicate Logic*.
- But first a recap on Semantic Tableau for *Propositional Logic*!

Recap: Semantic Tableau

- Today, we cover Semantic Tableau for *Predicate Logic*.
- But first a recap on Semantic Tableau for *Propositional Logic*!
- If we want to prove $X \vdash A$ (with $X = \{A_1, \dots, A_n\}$), then, we:
 - Label each assumption A_1, \dots, A_n as being *true* (**T**),
 - Label A as being *false* (**F**),

Recap: Semantic Tableau

- Today, we cover Semantic Tableau for *Predicate Logic*.
- But first a recap on Semantic Tableau for *Propositional Logic*!
- If we want to prove $X \vdash A$ (with $X = \{A_1, \dots, A_n\}$), then, we:
 - Label each assumption A_1, \dots, A_n as being *true* (**T**),
 - Label A as being *false* (**F**),
 - Simplify each formula (according to the connectives corresponding to truth tables) thus eventually obtaining:
 - 1 a contradiction in all the branches, or
 - 2 ≥ 1 open branch (i.e., none of its formulae can be simplified further and there's no contradiction).

In case 1 the sequent is *valid*.

In case 2 the sequent is *invalid*, and we can construct an interpretation that makes all formulae in X true, but A false (which is a witness for invalidity).

Recap: Example for a Valid Sequent

$$p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$$

(1) **T:** $p \rightarrow q$

(2) **T:** $r \rightarrow s$

(3) **F:** $(p \vee r) \rightarrow (q \vee s)$

$$\frac{\mathbf{T}: A \rightarrow B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$$

Recap: Example for a Valid Sequent

$$p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$$

(1) **T:** $p \rightarrow q$

(2) **T:** $r \rightarrow s$

(3) **F:** $(p \vee r) \rightarrow (q \vee s)$ ✓

(4) **T:** $p \vee r$

from (3)

(5) **F:** $q \vee s$

from (3)

$$\frac{\mathbf{T}: A \rightarrow B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$$

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \vee B}{\mathbf{F}: A, \mathbf{F}: B}$$

Recap: Example for a Valid Sequent

$$p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$$

(1) **T:** $p \rightarrow q$

(2) **T:** $r \rightarrow s$

(3) **F:** $(p \vee r) \rightarrow (q \vee s)$ ✓

(4) **T:** $p \vee r$

from (3)

(5) **F:** $q \vee s$ ✓

from (3)

(6) **F:** q

from (5)

(7) **F:** s

from (5)

$$\frac{\mathbf{T}: A \rightarrow B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$$

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \vee B}{\mathbf{F}: A, \mathbf{F}: B}$$

Recap: Example for a Valid Sequent

$$p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$$

(1) **T:** $p \rightarrow q$ ✓

(2) **T:** $r \rightarrow s$

(3) **F:** $(p \vee r) \rightarrow (q \vee s)$ ✓

(4) **T:** $p \vee r$ from (3)

(5) **F:** $q \vee s$ ✓ from (3)

(6) **F:** q from (5)

(7) **F:** s from (5)

(8) **F:** p from (1)

(9) **T:** q ✗ from (1)

$$\frac{\mathbf{T}: A \rightarrow B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$$

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \vee B}{\mathbf{F}: A, \mathbf{F}: B}$$

Recap: Example for a Valid Sequent

$$p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$$

(1) **T:** $p \rightarrow q$ ✓

(2) **T:** $r \rightarrow s$ ✓

(3) **F:** $(p \vee r) \rightarrow (q \vee s)$ ✓

(4) **T:** $p \vee r$ from (3)

(5) **F:** $q \vee s$ ✓ from (3)

(6) **F:** q from (5)

(7) **F:** s from (5)

(8) **F:** p from (1)

(9) **T:** q ✗ from (1)

(10) **F:** r from (2)

(11) **T:** s ✗ from (2)

$$\frac{\mathbf{T}: A \rightarrow B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$$

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \vee B}{\mathbf{F}: A, \mathbf{F}: B}$$

Recap: Example for a Valid Sequent

$$p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$$

(1) **T:** $p \rightarrow q$ ✓

(2) **T:** $r \rightarrow s$ ✓

(3) **F:** $(p \vee r) \rightarrow (q \vee s)$ ✓

(4) **T:** $p \vee r$ ✓ from (3)

(5) **F:** $q \vee s$ ✓ from (3)

(6) **F:** q from (5)

(7) **F:** s from (5)

(8) **F:** p from (1)

(9) **T:** q ✗ from (1)

(10) **F:** r from (2)

(11) **T:** s ✗ from (2)

(12) **T:** p ✗ from (4)

(13) **T:** r ✗ from (4)

$$\frac{\mathbf{T}: A \rightarrow B}{\mathbf{F}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \rightarrow B}{\mathbf{T}: A, \mathbf{F}: B}$$

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \vee B}{\mathbf{F}: A, \mathbf{F}: B}$$

Recap: Example for an Invalid Sequent

$$p \vee q \vdash p \wedge q$$

$$(1) \quad \mathbf{T}: \quad p \vee q$$

$$(2) \quad \mathbf{F}: \quad p \wedge q$$

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \wedge B}{\mathbf{F}: A \mid \mathbf{F}: B}$$

Recap: Example for an Invalid Sequent

$$p \vee q \vdash p \wedge q$$

(1) **T:** $p \vee q$ ✓

(2) **F:** $p \wedge q$

(3) **T:** p from (1)

(4) **T:** q from (1)

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$
$$\frac{\mathbf{F}: A \wedge B}{\mathbf{F}: A \mid \mathbf{F}: B}$$

Recap: Example for an Invalid Sequent

$$p \vee q \vdash p \wedge q$$

(1) **T:** $p \vee q$ ✓

(2) **F:** $p \wedge q$ (✓)

(3) **T:** p from (1)

(4) **T:** q from (1)

(5) **F:** p ⚡ from (2)

(6) **F:** q *open!* from (2)

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \wedge B}{\mathbf{F}: A \mid \mathbf{F}: B}$$

Recap: Example for an Invalid Sequent

$$p \vee q \vdash p \wedge q$$

(1) **T:** $p \vee q$ ✓

(2) **F:** $p \wedge q$ (✓)

(3) **T:** p from (1)

(4) **T:** q from (1)

(5) **F:** p ⚡ from (2)

(6) **F:** q *open!* from (2)

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\frac{\mathbf{F}: A \wedge B}{\mathbf{F}: A \mid \mathbf{F}: B}$$

- We've found an open branch, so we are allowed to stop the proof!
- The interpretation $I_1(p) = \top$ and $I_1(q) = \perp$ proves that the assumption $p \vee q$ can be made true while the formula $p \wedge q$ is false, thus invalidating the sequent.

Recap: Example for an Invalid Sequent

$$p \vee q \vdash p \wedge q$$

$$(1) \quad \mathbf{T}: p \vee q \quad \checkmark$$

$$(2) \quad \mathbf{F}: p \wedge q \quad (\checkmark)$$

$$(3) \quad \mathbf{T}: p \quad \text{from (1)}$$

$$(4) \quad \mathbf{T}: q \quad \text{from (1)}$$

$$(5) \quad \mathbf{F}: p \quad \text{from (2)}$$

$$(6) \quad \mathbf{F}: q \quad \text{open!} \quad \text{from (2)}$$

$$\frac{\mathbf{T}: A \vee B}{\mathbf{T}: A \mid \mathbf{T}: B}$$

$$\mathbf{T}: A \mid \mathbf{T}: B$$

$$\frac{\mathbf{F}: A \wedge B}{\mathbf{F}: A \mid \mathbf{F}: B}$$

$$\mathbf{F}: A \mid \mathbf{F}: B$$

- We've found an open branch, so we are allowed to stop the proof!
- The interpretation $I_1(p) = \top$ and $I_1(q) = \perp$ proves that the assumption $p \vee q$ can be made true while the formula $p \wedge q$ is false, thus invalidating the sequent.
- But we could also expand the right branch (line (4) due to line (2)) to obtain the second interpretation $I_2(p) = \perp$ and $I_2(q) = \top$. (Not required unless we explicitly ask you to.)

Today: Properties of Logics and Proof Systems

Properties of Logics:

- What does it mean to *decide* validity?
- Is that always possible for sequents in Propositional Logic?
What about Predicate Logic?

Today: Properties of Logics and Proof Systems

Properties of Logics:

- What does it mean to *decide* validity?
- Is that always possible for sequents in Propositional Logic?
What about Predicate Logic?

Properties of Proof Systems:

- Are all proofs correct? (Soundness)
- Can we always prove validity? (Completeness)

Today: Semantic Tableau for Predicate Logic

- We still use the same rules as we had in the propositional case.
- But now we introduce four additional rules, namely for:
 - \exists -formulae which are labeled *true*
 - ... *false*
 - \forall -formulae which are labeled *true*
 - ... *false*

Today: Semantic Tableau for Predicate Logic

- We still use the same rules as we had in the propositional case.
- But now we introduce four additional rules, namely for:
 - \exists -formulae which are labeled *true*
 - ... *false*
 - \forall -formulae which are labeled *true*
 - ... *false*
- We also introduce additional rules for the special case that we want to prove invalidity:
 - \exists -formulae which are labeled *true*
 - \forall -formulae ... *false*
 - They will (sometimes) help to prove (more) sequents as invalid!

Properties

Recap on our Symbols and their Meanings

We differentiate between *validity* and *provability*:

- $X \models A$ (A follows logically from X)
 - Every interpretation that makes X true also makes A true.
- $X \vdash_{ND} A$ ($X \vdash A$ can be proved via *Natural Deduction*)
 - A can be *derived* from X . (Syntax manipulation.)
- $X \vdash_{ST} A$ ($X \vdash A$ can be proved via *Semantic Tableau*)
 - We can't find an interpretation that makes X true but not A . (Exploits validity definition.)
- There are many more proof systems!

Syntax vs. Semantics

- So, what's the relation between $X \models A$ and $X \vdash A$?
- A desirable situation would be $X \models A$ iff $X \vdash A$.
- Our proof systems could do *anything!* So what *could* happen?

Syntax vs. Semantics

- So, what's the relation between $X \models A$ and $X \vdash A$?
- A desirable situation would be $X \models A$ iff $X \vdash A$.
- Our proof systems could do *anything*! So what *could* happen?
- Let \mathcal{X} be *some* proof system (like, e.g., ND).
 - 1 $X \vdash_{\mathcal{X}} A$, but not $X \models A$
 - The proof system is wrong! (i.e., not sound.)

Syntax vs. Semantics

- So, what's the relation between $X \models A$ and $X \vdash A$?
- A desirable situation would be $X \models A$ iff $X \vdash A$.
- Our proof systems could do *anything*! So what *could* happen?
- Let \mathcal{X} be *some* proof system (like, e.g., ND).
 - 1 $X \vdash_{\mathcal{X}} A$, but not $X \models A$
 - The proof system is wrong! (i.e., not sound.)
 - 2 $X \models A$, but not $X \vdash_{\mathcal{X}} A$
 - The proof system is incomplete! (i.e., not complete.)

Syntax vs. Semantics

- So, what's the relation between $X \models A$ and $X \vdash A$?
- A desirable situation would be $X \models A$ iff $X \vdash A$.
- Our proof systems could do *anything*! So what *could* happen?
- Let \mathcal{X} be *some* proof system (like, e.g., ND).
 - 1 $X \vdash_{\mathcal{X}} A$, but not $X \models A$
 - The proof system is wrong! (i.e., not sound.)
 - 2 $X \models A$, but not $X \vdash_{\mathcal{X}} A$
 - The proof system is incomplete! (i.e., not complete.)
- What we want:
 - Soundness Every provable sequent is valid. (Cf. above's 1)
 - Completeness Every valid sequent is provable. (Cf. above's 2)

Decision Procedure

- Let \mathcal{X} be some proof system that's sound and complete.
- So, can we also *decide* validity of each sequent with \mathcal{X} ?
- I.e., we want to know whether $X \models A$ holds, by using \mathcal{X} .
Can we find out?
- Again, \mathcal{X} is sound and complete, so we can check validity, right?

Decision Procedure

- Let \mathcal{X} be some proof system that's sound and complete.
- So, can we also *decide* validity of each sequent with \mathcal{X} ?
- I.e., we want to know whether $X \models A$ holds, by using \mathcal{X} .
Can we find out?
- Again, \mathcal{X} is sound and complete, so we can check validity, right?
- No, not necessarily! Both just mention validity, not invalidity!
- We only know: $X \models A$ iff $X \vdash A$
- But we don't necessarily know whether $X \models A$ holds since a sequent could also be invalid! (In that case maybe the proof system just keeps running... So we don't get $X \vdash A$, but we also don't get an output saying " $X \models A$ is false")

Properties of Logics and proof systems

Decidability of Logics:

- *Decidability of a Logic* means determining for an arbitrary sequent whether it's valid or not.
- Propositional Logic:
- Predicate Logic:

Properties of Logics and proof systems

Decidability of Logics:

- *Decidability of a Logic* means determining for an arbitrary sequent whether it's valid or not.
- Propositional Logic: Yes, decidable.
- Predicate Logic: No, undecidable. No such algorithm can exist.

Properties of Logics and proof systems

Decidability of Logics:

- *Decidability of a Logic* means determining for an arbitrary sequent whether it's valid or not.
- Propositional Logic: Yes, decidable.
- Predicate Logic: No, undecidable. No such algorithm can exist.

Soundness and Completeness of Proof Systems:

- Natural Deduction:
- Semantic Tableau:

Properties of Logics and proof systems

Decidability of Logics:

- *Decidability of a Logic* means determining for an arbitrary sequent whether it's valid or not.
- Propositional Logic: Yes, decidable.
- Predicate Logic: No, undecidable. No such algorithm can exist.

Soundness and Completeness of Proof Systems:

- Natural Deduction:
 - Sound and complete for Propositional and Predicate Logic
- Semantic Tableau:
 - Also Sound and complete for Propositional and Predicate Logic

Semantic Tableau Rules

Simplifying a *true* \exists Quantifier (Intuition)

$$\frac{\mathbf{T}: \exists x Fx}{\mathbf{T}: Fa} \quad \text{provided } a \text{ is new to the branch}$$

- Why does *a* need to be new?
- Think of the triangle ABC! If *a* would exist already in the branch it would not be general (e.g., we could “accidentally” assume that our triangle is rectangular).

Simplifying a *false* \forall Quantifier (Intuition)

$$\frac{\mathbf{F}: \forall x Fx}{\mathbf{F}: Fa} \quad \text{provided } a \text{ is new to the branch}$$

- This corresponds to the true existential quantifier!
- Recall $\neg \forall x Fx \equiv \exists x \neg Fx$

Rules For *true* \exists and *false* \forall , formally
$$\mathbf{T: \exists x Fx}$$
$$\mathbf{T: Fa}$$

*if a is new to
the branch*

$$\equiv$$
$$X, \mathbf{T: \exists x A}$$
$$X, \mathbf{T: A_x^a}$$

for a not in X or A

Rules For *true* \exists and *false* \forall , formally

$$\frac{\mathbf{T}: \exists x Fx}{\mathbf{T}: Fa}$$

if a is new to the branch

$$\equiv$$

$$\frac{X, \mathbf{T}: \exists x A}{X, \mathbf{T}: A_x^a}$$

for a not in X or A

- The X represents all other lines we have in that branch.
- This notation shows nicely why we use checkmarks:
 - They show which lines have been processed.
 - Here it's those lines which don't appear below the line anymore! Note that $\mathbf{T}: \exists x A$ doesn't appear anymore, meaning that the rule is "processed" already.

Rules For *true* \exists and *false* \forall , formally

$$\mathbf{T: \exists x Fx}$$

$$\mathbf{T: Fa}$$

*if a is new to
the branch*

$$\mathbf{F: \forall x Fx}$$

$$\mathbf{F: Fa}$$

*if a is new to
the branch*

$$\equiv$$

$$\equiv$$

$$X, \mathbf{T: \exists x A}$$

$$X, \mathbf{T: A_x^a}$$

for a not in X or A

$$X, \mathbf{F: \forall x A}$$

$$X, \mathbf{F: A_x^a}$$

for a not in X or A

Simplifying a *true* \forall Quantifier (Intuition)

T: $\forall x Fx$

T: $Fa, T: Fb, \dots$ for all a, b, \dots in the branch (present and future!)

- This rule will continue being available for new constants/terms produced later on. (Then we have to apply the rule again!)
- If we already obtained a contradiction, we are clearly done. But if we want to show that a branch is open we need to have applied this rule to *all* constants! (I.e., also those that get created after we already applied the rule to all constants that existed until then.)

Simplifying a *false* \exists Quantifier (Intuition)

$\mathbf{F}: \exists x Fx$

 $\mathbf{F}: Fa, \mathbf{F}: Fb, \dots$ for all a, b, \dots in the branch (present and future!)

- Again, this rule will never be finished! If a new constant/term gets introduced we need to apply the rule again!
- Recall from last week that $\neg \exists x Fx \equiv \forall x \neg Fx$

Rules for *true* \forall and *false* \exists , formally
$$\frac{\mathbf{T}: \forall x Fx}{\mathbf{T}: Fa, \mathbf{T}: Fb, \dots}$$

*for all a, b, ...
in the branch –
present and future!*

≡

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

$$\frac{\mathbf{F}: \exists x Fx}{\mathbf{F}: Fa, \mathbf{F}: Fb, \dots}$$

*for all a, b, ...
in the branch –
present and future!*

≡

$$\frac{X, \mathbf{F}: \exists x A}{X, \mathbf{F}: \exists x A, \mathbf{F}: A_x^a}$$

for a in X or A

Special case for false Existential and true Universal

- Recall the rules for false existentials and true universals:

$$\frac{\mathbf{F}: \exists x Fx}{\mathbf{F}: Fa, \mathbf{F}: Fb, \dots}$$

*for all a, b, ...
in the branch –
present and future!*

$$\frac{\mathbf{T}: \forall x Fx}{\mathbf{T}: Fa, \mathbf{T}: Fb, \dots}$$

*for all a, b, ...
in the branch –
present and future!*

- They state that you only “use” constants which are already there.
- Sometimes, however, there one no such constants! Then, you are also allowed to create a new one.
- Do you need an example? Create one!

Examples

Quick Note

In the following examples we provide the rules in their non-general form (that uses an example!) – this just done for didactic reasons as this notation is easier to grasp.

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$
 (2) **F:** $\forall x Fx \vee \forall x Gx$

T: $\forall x Fx$

T: $Fa, \mathbf{T: Fb}, \dots$

*for all a, b, ...
 in the branch –
 present and future!*

F: $\forall x Fx$

F: Fa

*if a is new to
 the branch*

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$
 (2) **F:** $\forall x Fx \vee \forall x Gx$ ✓
 (3) **F:** $\forall x Fx$ from (2)
 (4) **F:** $\forall x Gx$ from (2)

T: $\forall x Fx$ <hr/> T: $Fa, T: Fb, \dots$ <i>for all a, b, ...</i> <i>in the branch –</i> <i>present and future!</i>
--

Trick question! :) (In the live lecture...)

Note that we did not apply the rule for false universal quantifier here because the formula is actually a false *disjunction*, not a false universally quantified formula.

F: $\forall x Fx$ <hr/> F: Fa <i>if a is new to</i> <i>the branch</i>
--

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$
 (2) **F:** $\forall x Fx \vee \forall x Gx$ ✓
 (3) **F:** $\forall x Fx$ ✓ from (2)
 (4) **F:** $\forall x Gx$ from (2)
 (5) **F:** Fa from (3)

T: $\forall x Fx$

T: $Fa, \mathbf{T: Fb}, \dots$

*for all a, b, ...
in the branch –
present and future!*

F: $\forall x Fx$

F: Fa

*if a is new to
the branch*

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$
 (2) **F:** $\forall x Fx \vee \forall x Gx$ ✓
 (3) **F:** $\forall x Fx$ ✓ from (2)
 (4) **F:** $\forall x Gx$ ✓ from (2)
 (5) **F:** Fa from (3)
 (6) **F:** Gb from (4)

T: $\forall x Fx$

T: $Fa, \mathbf{T: Fb}, \dots$

*for all a, b, ...
in the branch –
present and future!*

F: $\forall x Fx$

F: Fa

*if a is new to
the branch*

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$ ✓^a
 (2) **F:** $\forall x Fx \vee \forall x Gx$ ✓
 (3) **F:** $\forall x Fx$ ✓ from (2)
 (4) **F:** $\forall x Gx$ ✓ from (2)
 (5) **F:** Fa from (3)
 (6) **F:** Gb from (4)
 (7) **T:** $Fa \vee Ga$ from (1)

T: $\forall x Fx$

T: $Fa, \mathbf{T: Fb}, \dots$

*for all a, b, ...
in the branch –
present and future!*

F: $\forall x Fx$

F: Fa

*if a is new to
the branch*

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$ ✓^{a,b}
 (2) **F:** $\forall x Fx \vee \forall x Gx$ ✓
 (3) **F:** $\forall x Fx$ ✓ from (2)
 (4) **F:** $\forall x Gx$ ✓ from (2)
 (5) **F:** Fa from (3)
 (6) **F:** Gb from (4)
 (7) **T:** $Fa \vee Ga$ from (1)
 (8) **T:** $Fb \vee Gb$ from (1)

$$\text{T: } \forall x Fx$$

$$\text{T: } Fa, \text{T: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{F: } \forall x Fx$$

$$\text{F: } Fa$$

*if a is new to
the branch*

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$ ✓^{a,b}
- (2) **F:** $\forall x Fx \vee \forall x Gx$ ✓
- (3) **F:** $\forall x Fx$ ✓ from (2)
- (4) **F:** $\forall x Gx$ ✓ from (2)
- (5) **F:** Fa from (3)
- (6) **F:** Gb from (4)
- (7) **T:** $Fa \vee Ga$ ✓ from (1)
- (8) **T:** $Fb \vee Gb$ from (1)
- (9) **T:** Fa ⚡ from (7)
- (10) **T:** Ga from (7)

T: $\forall x Fx$

T: $Fa, \mathbf{T: Fb}, \dots$

*for all a, b, ...
in the branch –
present and future!*

F: $\forall x Fx$

F: Fa

*if a is new to
the branch*

Example 1

$$\forall x(Fx \vee Gx) \vdash? \forall x Fx \vee \forall x Gx$$

- (1) **T:** $\forall x(Fx \vee Gx)$ ✓^{a,b}
 (2) **F:** $\forall x Fx \vee \forall x Gx$ ✓
 (3) **F:** $\forall x Fx$ ✓ from (2)
 (4) **F:** $\forall x Gx$ ✓ from (2)
 (5) **F:** Fa from (3)
 (6) **F:** Gb from (4)
 (7) **T:** $Fa \vee Ga$ ✓ from (1)
 (8) **T:** $Fb \vee Gb$ ✓ from (1)

- (9) **T:** Fa ⚡ from (7) (10) **T:** Ga from (7)

- (11) **T:** Fb **open!** from (8) (12) **T:** Gb ⚡ from (8)

$$\text{T: } \forall x Fx$$

$$\text{T: } Fa, \text{T: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{F: } \forall x Fx$$

$$\text{F: } Fa$$

*if a is new to
the branch*

Extracted interpretation: see next slide.

Example 1 (cont'd)

So? Is $\forall x(Fx \vee Gx) \vdash \forall x Fx \vee \forall x Gx$ valid?

- Let's see... Not all branches are contradictory.
- Thus, there is an open branch:

(5) **F:** Fa from (3)

(6) **F:** Gb from (4)

(10) **T:** Ga from (7)

(11) **T:** Fb from (8)

- We can, as usual, extract an *interpretation* I that answers for which objects F and G is true:

- Informally: $I(Fa) = \perp$ and $I(Fb) = \top$ The formal definition will
 - Informally: $I(Ga) = \top$ and $I(Gb) = \perp$ be provided in week 7
 - Thus, showing that there is an interpretation that makes the assumption true, but the formula false!
- So the sequent is invalid!

Example 1 (cont'd, once more!)

What's the *intuitive* way of interpreting the invalidity of

$\forall x(Fx \vee Gx) \vdash \forall x Fx \vee \forall x Gx$?

- There's a “world” in which the the sequent doesn't hold!
- The sequent would be valid if in all “worlds” it holds:
 - If all “individuals” are footballers or goats,
 - then all individuals are footballers or all individuals are goats.

Example 1 (cont'd, once more!)

What's the *intuitive* way of interpreting the invalidity of

$\forall x(Fx \vee Gx) \vdash \forall x Fx \vee \forall x Gx$?

- There's a "world" in which the the sequent doesn't hold!
- The sequent would be valid if in all "worlds" it holds:
 - If all "individuals" are footballers or goats,
 - then all individuals are footballers or all individuals are goats.
- It certainly *can* be true!
 - If there's just one individual! Let's call it Pascal and assume he/it is a footballer, a goat, or both. Then the sequent is valid!

Example 1 (cont'd, once more!)

What's the *intuitive* way of interpreting the invalidity of

$\forall x(Fx \vee Gx) \vdash \forall x Fx \vee \forall x Gx$?

- There's a “world” in which the the sequent doesn't hold!
- The sequent would be valid if in all “worlds” it holds:
 - If all “individuals” are footballers or goats,
 - then all individuals are footballers or all individuals are goats.
- It certainly *can* be true!
 - If there's just one individual! Let's call it Pascal and assume he/it is a footballer, a goat, or both. Then the sequent is valid!
 - Let a and b be footballers, none of them is a goat. Still valid!

Example 1 (cont'd, once more!)

What's the *intuitive* way of interpreting the invalidity of

$\forall x(Fx \vee Gx) \vdash \forall x Fx \vee \forall x Gx$?

- There's a "world" in which the the sequent doesn't hold!
- The sequent would be valid if in all "worlds" it holds:
 - If all "individuals" are footballers or goats,
 - then all individuals are footballers or all individuals are goats.
- It certainly *can* be true!
 - If there's just one individual! Let's call it Pascal and assume he/it is a footballer, a goat, or both. Then the sequent is valid!
 - Let a and b be footballers, none of them is a goat. Still valid!
- But in our counter-example world it's not true!
 - a is a goat, b is a footballer. So everything is a footballer or a goat.
 - But it's not true that a and b are footballers or a and b are goats!

Example 2

$$\exists x Fx, \exists x Gx \vdash^? \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$
 (2) **T:** $\exists x Gx$
 (3) **F:** $\exists x (Fx \wedge Gx)$

$$\mathbf{T:} \exists x Fx$$

$$\mathbf{T:} Fa$$

*if a is new to
the branch*

$$\mathbf{F:} \exists x Fx$$

$$\mathbf{F:} Fa, \mathbf{F:} Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

Example 2

$$\exists x Fx, \exists x Gx \vdash? \exists x (Fx \wedge Gx)$$
(1) **T:** $\exists x Fx$ ✓(2) **T:** $\exists x Gx$ (3) **F:** $\exists x (Fx \wedge Gx)$ (4) **T:** Fa

from (1)

T: $\exists x Fx$ **T:** Fa *if a is new to
the branch***F:** $\exists x Fx$ **F:** $Fa, \mathbf{F: Fb}, \dots$ *for all a, b, ...
in the branch –
present and future!*

Example 2

$$\exists x Fx, \exists x Gx \vdash? \exists x (Fx \wedge Gx)$$
(1) **T:** $\exists x Fx$ ✓(2) **T:** $\exists x Gx$ ✓(3) **F:** $\exists x (Fx \wedge Gx)$ (4) **T:** Fa from (1)(5) **T:** Gb from (2)**T:** $\exists x Fx$ **T:** Fa *if a is new to
the branch***F:** $\exists x Fx$ **F:** $Fa, \mathbf{F: Fb}, \dots$ *for all a, b, ...
in the branch –
present and future!*

Example 2

$$\exists x Fx, \exists x Gx \vdash^? \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$ ✓
 (2) **T:** $\exists x Gx$ ✓
 (3) **F:** $\exists x (Fx \wedge Gx)$ ✓^a
 (4) **T:** Fa from (1)
 (5) **T:** Gb from (2)
 (6) **F:** $Fa \wedge Ga$ from (3)

$$\text{T: } \exists x Fx$$

$$\text{T: } Fa$$

*if a is new to
the branch*

$$\text{F: } \exists x Fx$$

$$\text{F: } Fa, \text{F: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

Example 2

$$\exists x Fx, \exists x Gx \vdash^? \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$ ✓
 (2) **T:** $\exists x Gx$ ✓
 (3) **F:** $\exists x (Fx \wedge Gx)$ ✓^{a,b}
 (4) **T:** Fa from (1)
 (5) **T:** Gb from (2)
 (6) **F:** $Fa \wedge Ga$ from (3)
 (7) **F:** $Fb \wedge Gb$ from (3)

$$\mathbf{T:} \exists x Fx$$

$$\mathbf{T:} Fa$$

*if a is new to
the branch*

$$\mathbf{F:} \exists x Fx$$

$$\mathbf{F:} Fa, \mathbf{F:} Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

Example 2

$$\exists x Fx, \exists x Gx \vdash^? \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$ ✓
 (2) **T:** $\exists x Gx$ ✓
 (3) **F:** $\exists x (Fx \wedge Gx)$ ✓^{a,b}
 (4) **T:** Fa from (1)
 (5) **T:** Gb from (2)
 (6) **F:** $Fa \wedge Ga$ ✓ from (3)
 (7) **F:** $Fb \wedge Gb$ from (3)
- (8) **F:** Fa ⚡ from (6) (9) **F:** Ga from (6)

$$\mathbf{T:} \exists x Fx$$

$$\mathbf{T:} Fa$$

*if a is new to
the branch*

$$\mathbf{F:} \exists x Fx$$

$$\mathbf{F:} Fa, \mathbf{F:} Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

Example 2

$$\exists x Fx, \exists x Gx \vdash^? \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$ ✓
 (2) **T:** $\exists x Gx$ ✓
 (3) **F:** $\exists x (Fx \wedge Gx)$ ✓^{a,b}
 (4) **T:** Fa from (1)
 (5) **T:** Gb from (2)
 (6) **F:** $Fa \wedge Ga$ ✓ from (3)
 (7) **F:** $Fb \wedge Gb$ ✓ from (3)

(8) **F:** Fa ⚡ from (6) (9) **F:** Ga from (6)

(10) **F:** Fb *open!* from (7) (11) **F:** Gb ⚡ from (7)

$$\frac{\mathbf{T}: \exists x Fx}{\mathbf{T}: Fa}$$

$$\mathbf{T}: Fa$$

*if a is new to
the branch*

$$\frac{\mathbf{F}: \exists x Fx}{\mathbf{F}: Fa, \mathbf{F}: Fb, \dots}$$

$$\mathbf{F}: Fa, \mathbf{F}: Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

Extracted interpretation: see next slide.

Example 2 (cont'd)

So? Is $\exists x Fx, \exists x Gx \vdash^? \exists x (Fx \wedge Gx)$ valid?

- Let's see... Not all branches are contradictory.
- Thus, there is an open branch:

(4) **T:** Fa from (1)

(5) **T:** Gb from (2)

(9) **F:** Ga from (6)

(10) **F:** Fb from (7)

- Again we can design an interpretation that answers for which objects F and G become true:
 - F is true for exactly a
 - G is true for exactly b
 - Thus, showing that there is an interpretation that makes the assumption true, but the formula false!
- So the sequent is invalid!

Example by de'Morgan

Intended to show how Predicate Logic goes beyond Propositional Logic:

- All horses are animals.
- Therefore, any horse's head is an animal head!

Example by de'Morgan

Intended to show how Predicate Logic goes beyond Propositional Logic:

- All horses are animals.
- Therefore, any horse's head is an animal head!

We formalize this in terms of Predicate Logic.

- Instead of:
“any horse's head is an animal head”
- We formalize that as: “each part of a horse is part of an animal”
$$\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

Example by de'Morgan

Intended to show how Predicate Logic goes beyond Propositional Logic:

- All horses are animals.
- Therefore, any horse's head is an animal head!

We formalize this in terms of Predicate Logic.

- Instead of:
“any horse's head is an animal head”
- We formalize that as: “each part of a horse is part of an animal”
$$\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

Thus we get:

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

$$(1) \quad \mathbf{T}: \quad \forall x Hx \rightarrow Ax$$

$$(2) \quad \mathbf{F}: \quad \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

$$\mathbf{T}: \forall x Fx$$

$$\mathbf{T}: Fa, \mathbf{T}: Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\mathbf{F}: \forall x Fx$$

$$\mathbf{F}: Fa$$

*if a is new to
the branch*

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ from (2)

$$\mathbf{T:} \forall x Fx$$

$$\mathbf{T:} Fa, \mathbf{T:} Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\mathbf{F:} \forall x Fx$$

$$\mathbf{F:} Fa$$

*if a is new to
the branch*

$$\mathbf{F:} \exists x Fx$$

$$\mathbf{F:} Fa, \mathbf{F:} Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ from (3)

$$\text{T: } \forall x Fx$$

$$\text{T: } Fa, \text{T: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{F: } \forall x Fx$$

$$\text{F: } Fa$$

*if a is new to
the branch*

$$\text{F: } \exists x Fx$$

$$\text{F: } Fa, \text{F: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{T: } \exists x Fx$$

$$\text{T: } Fa$$

*if a is new to
the branch*

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ from (3)
- (6) **T:** $Hb \wedge Pab$ from (4)

$$\text{T: } \forall x Fx$$

$$\text{T: } Fa, \text{T: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{F: } \forall x Fx$$

$$\text{F: } Fa$$

*if a is new to
the branch*

$$\text{F: } \exists x Fx$$

$$\text{F: } Fa, \text{F: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{T: } \exists x Fx$$

$$\text{T: } Fa$$

*if a is new to
the branch*

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)

$$\text{T: } \forall x Fx$$

$$\text{T: } Fa, \text{T: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{F: } \forall x Fx$$

$$\text{F: } Fa$$

*if a is new to
the branch*

$$\text{F: } \exists x Fx$$

$$\text{F: } Fa, \text{F: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{T: } \exists x Fx$$

$$\text{T: } Fa$$

*if a is new to
the branch*

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$ ✓^b
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **T:** $Hb \rightarrow Ab$ from (1)

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$ ✓^b
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **T:** $Hb \rightarrow Ab$ ✓ from (1)

(10) **F:** Hb ⚡ from (9)

(11) **T:** Ab from (9)

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$ ✓^b
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ ✓^b from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **T:** $Hb \rightarrow Ab$ ✓ from (1)

(10) **F:** Hb ⚡ from (9)

(11) **T:** Ab from (9)

(12) **F:** $Ab \wedge Pab$ from (5)

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$ ✓^b
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ ✓^b from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **T:** $Hb \rightarrow Ab$ ✓ from (1)

- (10) **F:** Hb ⚡ from (9)
- (11) **T:** Ab from (9)
- (12) **F:** $Ab \wedge Pab$ ✓ from (5)

- (13) **F:** Ab ⚡ from (12)
- (14) **F:** Pab ⚡ from (12)

Example 3

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$ ✓^b
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ ✓^b from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **T:** $Hb \rightarrow Ab$ ✓ from (1)

- (10) **F:** Hb ⚡ from (9)
- (11) **T:** Ab from (9)
- (12) **F:** $Ab \wedge Pab$ ✓ from (5)

- (13) **F:** Ab ⚡ from (12)
- (14) **F:** Pab ⚡ from (12)

All branches are contradictory. Sequent is *valid*!

Example 3 (Again with a different Order)

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- | | | | |
|-----|-----------|--|------------|
| (1) | T: | $\forall x Hx \rightarrow Ax$ | |
| (2) | F: | $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ | ✓ |
| (3) | F: | $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ | ✓ from (2) |
| (4) | T: | $\exists y(Hy \wedge Pay)$ | ✓ from (3) |
| (5) | F: | $\exists y(Ay \wedge Pay)$ | from (3) |
| (6) | T: | $Hb \wedge Pab$ | ✓ from (4) |
| (7) | T: | Hb | from (6) |
| (8) | T: | Pab | from (6) |

$$\text{T: } \forall x Fx$$

$$\text{T: } Fa, \text{T: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{F: } \forall x Fx$$

$$\text{F: } Fa$$

*if a is new to
the branch*

$$\text{F: } \exists x Fx$$

$$\text{F: } Fa, \text{F: } Fb, \dots$$

*for all a, b, ...
in the branch –
present and future!*

$$\text{T: } \exists x Fx$$

$$\text{T: } Fa$$

*if a is new to
the branch*

Example 3 (Again with a different Order)

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- | | | | |
|-----|-----------|--|-------------------------|
| (1) | T: | $\forall x Hx \rightarrow Ax$ | |
| (2) | F: | $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ | ✓ |
| (3) | F: | $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ | ✓ from (2) |
| (4) | T: | $\exists y(Hy \wedge Pay)$ | ✓ from (3) |
| (5) | F: | $\exists y(Ay \wedge Pay)$ | ✓ ^b from (3) |
| (6) | T: | $Hb \wedge Pab$ | ✓ from (4) |
| (7) | T: | Hb | from (6) |
| (8) | T: | Pab | from (6) |
| (9) | F: | $Ab \wedge Pab$ | from (5) |

Example 3 (Again with a different Order)

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ ✓^b from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **F:** $Ab \wedge Pab$ ✓ from (5)
- (10) **F:** Ab from (9)
- (11) **F:** Pab ⚡ from (9)

Example 3 (Again with a different Order)

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$ ✓^b
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ ✓^b from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **F:** $Ab \wedge Pab$ ✓ from (5)
- (10) **F:** Ab from (9) (11) **F:** Pab ⚡ from (9)
- (12) **T:** $Hb \rightarrow Ab$ from (1)

Example 3 (Again with a different Order)

$$\forall x Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$$

- (1) **T:** $\forall x Hx \rightarrow Ax$ ✓^b
- (2) **F:** $\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓
- (3) **F:** $\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓ from (2)
- (4) **T:** $\exists y(Hy \wedge Pay)$ ✓ from (3)
- (5) **F:** $\exists y(Ay \wedge Pay)$ ✓^b from (3)
- (6) **T:** $Hb \wedge Pab$ ✓ from (4)
- (7) **T:** Hb from (6)
- (8) **T:** Pab from (6)
- (9) **F:** $Ab \wedge Pab$ ✓ from (5)

- (10) **F:** Ab from (9) (11) **F:** Pab ⚡ from (9)
- (12) **T:** $Hb \rightarrow Ab$ ✓ from (1)

- (13) **F:** Hb ⚡ from (12) (14) **T:** Ab ⚡ from (12)

All branches are contradictory. Sequent is *valid!*

Invalid Sequents

Motivation: Sequent is invalid, so?

- You learned that there are some invalid sequents for which you can't find a proof that shows invalidity.
- (We were however still able to find invalidity proofs for *some* invalid sequents, e.g., examples 1 and 2.)
- In some cases however, we *could* prove invalidity if we had more clever rules – which are tailored for finding invalid proofs.
- Even with these rules, though, we still can't *always* prove invalidity. (Since Predicate Logic is undecidable.)

Motivation: Motivating Example

Assume we are deep within some branch:

(n) **T:** $\forall x \exists y Rxy$ from (k < n)

$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$ <p><i>for a in X or A</i></p>
--

Motivation: Motivating Example

Assume we are deep within some branch:

- (n) **T:** $\forall x \exists y Rxy$ ✓^a from (k < n)
- (n+1) **T:** $\exists y Ray$ from (n)

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

$$\frac{X, \mathbf{T}: \exists x A}{X, \mathbf{T}: A_x^a}$$

for a not in X or A

Motivation: Motivating Example

Assume we are deep within some branch:

- (n) **T:** $\forall x \exists y Rxy$ ✓^a from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

$$\frac{X, \mathbf{T}: \exists x A}{X, \mathbf{T}: A_x^a}$$

for a not in X or A

Motivation: Motivating Example

Assume we are deep within some branch:

- (n) **T:** $\forall x \exists y Rxy$ ✓^{a,b} from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)
- (n+3) **T:** $\exists y Rby$ from (n)

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

$$\frac{X, \mathbf{T}: \exists x A}{X, \mathbf{T}: A_x^a}$$

for a not in X or A

Motivation: Motivating Example

Assume we are deep within some branch:

- (n) **T:** $\forall x \exists y Rxy$ ✓^{a,b} from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)
- (n+3) **T:** $\exists y Rby$ ✓ from (n)
- (n+4) **T:** Rbc from (n+3)

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

$$\frac{X, \mathbf{T}: \exists x A}{X, \mathbf{T}: A_x^a}$$

for a not in X or A

Motivation: Motivating Example

Assume we are deep within some branch:

- (n) **T:** $\forall x \exists y Rxy$ ✓^{a,b,c} from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)
- (n+3) **T:** $\exists y Rby$ ✓ from (n)
- (n+4) **T:** Rbc from (n+3)
- (n+5) **T:** $\exists y Rcy$ from (n)

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

$$\frac{X, \mathbf{T}: \exists x A}{X, \mathbf{T}: A_x^a}$$

for a not in X or A

Motivation: Motivating Example

Assume we are deep within some branch:

- (n) T: $\forall x \exists y Rxy$ ✓^{a,b,c} from (k < n)
- (n+1) T: $\exists y Ray$ ✓ from (n)
- (n+2) T: Rab from (n+1)
- (n+3) T: $\exists y Rby$ ✓ from (n)
- (n+4) T: Rbc from (n+3)
- (n+5) T: $\exists y Rcy$ ✓ from (n)
- (n+6) T: Rcd from (n+5)

$$\frac{X, T: \forall x A}{X, T: \forall x A, T: A_x^a}$$

for a in X or A

$$\frac{X, T: \exists x A}{X, T: A_x^a}$$

for a not in X or A

Motivation: Motivating Example

Assume we are deep within some branch:

- (n) **T:** $\forall x \exists y Rxy$ ✓^{a,b,c} from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)
- (n+3) **T:** $\exists y Rby$ ✓ from (n)
- (n+4) **T:** Rbc from (n+3)
- (n+5) **T:** $\exists y Rcy$ ✓ from (n)
- (n+6) **T:** Rcd from (n+5)

- So we have an infinite branch!
- We will *never* be able to show that it is open.
- Is this what we want?
- I.e., *should* we be able to prove that the branch is open?

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

$$\frac{X, \mathbf{T}: \exists x A}{X, \mathbf{T}: A_x^a}$$

for a not in X or A

Motivation: Showing Invalidity

- Recall what an invalid sequent $X \vdash A$ means, and how we do it:
 - It means that we can find an interpretation (“special case”) where all premises (i.e., formulae in X) are true, but not the formula A . So the formula A is not a logical consequence.
 - We prove that by providing such a “special case” interpretation!
 - More precisely, we derive an interpretation that is consistent with all formulae in the current branch, thus showing that it’s not always contradictory!
- In order to achieve our goal, i.e., find an interpretation for such an infinite branch, we will have to alter one of our rules.

Rules & Example

Showing Invalidity (Example cont'd)

(n) **T:** $\forall x \exists y Rxy$ from $(k < n)$

$X, \mathbf{T}: \forall x A$
<hr/>
$X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a$
<i>for a in X or A</i>

Showing Invalidity (Example cont'd)

- (n) **T:** $\forall x \exists y Rxy$ ✓^a from (k < n)
- (n+1) **T:** $\exists y Ray$ from (n)

$X, \mathbf{T}: \forall x A$
<hr style="width: 80%; margin: auto;"/>
$X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a$
<i>for a in X or A</i>

<i>inv</i>	$\mathbf{T}: \exists x Fx$
<hr style="width: 80%; margin: auto;"/>	
$\mathbf{T}: Fa \mid \mathbf{T}: Fb \mid \dots \mid \mathbf{T}: Fn$	
<i>for all a, b, ... in the branch or n new to the branch</i>	

Showing Invalidity (Example cont'd)

- (n) **T:** $\forall x \exists y Rxy$ ✓^a from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)

$X, \mathbf{T}: \forall x A$
$X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a$
<i>for a in X or A</i>

<i>inv</i>	$\mathbf{T}: \exists x Fx$
$\mathbf{T}: Fa \mid \mathbf{T}: Fb \mid \dots \mid \mathbf{T}: Fn$	
<i>for all a, b, ... in the branch or n new to the branch</i>	

Showing Invalidity (Example cont'd)

- (n) **T:** $\forall x \exists y Rxy$ ✓^{a,b} from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)
- (n+3) **T:** $\exists y Rby$ from (n)

$X, \mathbf{T}: \forall x A$
$X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a$
<i>for a in X or A</i>

<i>inv</i>	$\mathbf{T}: \exists x Fx$
$\mathbf{T}: Fa \mid \mathbf{T}: Fb \mid \dots \mid \mathbf{T}: Fn$	
<i>for all a, b, ... in the branch or n new to the branch</i>	

Showing Invalidity (Example cont'd)

- (n) **T:** $\forall x \exists y Rxy$ ✓^{a,b} from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)
- (n+3) **T:** $\exists y Rby$ ✓ from (n)
- (n+4) **T:** Rba from (n+3)

$$\frac{X, \mathbf{T}: \forall x A}{X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a}$$

for a in X or A

- Now we've only used two objects, a and b , and got our interpretation: Rab and Rba are true.
- Is that allowed for the purpose of finding an open branch?

$$\frac{\text{inv} \quad \mathbf{T}: \exists x Fx}{\mathbf{T}: Fa \mid \mathbf{T}: Fb \mid \dots \mid \mathbf{T}: Fn}$$

for all a, b, \dots in the branch
or n new to the branch

Showing Invalidity (Example cont'd)

- (n) **T:** $\forall x \exists y Rxy$ ✓^{a,b} from (k < n)
- (n+1) **T:** $\exists y Ray$ ✓ from (n)
- (n+2) **T:** Rab from (n+1)
- (n+3) **T:** $\exists y Rby$ ✓ from (n)
- (n+4) **T:** Rba from (n+3)

$X, \mathbf{T}: \forall x A$
<hr style="width: 80%; margin: auto;"/>
$X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a$
<i>for a in X or A</i>

- Now we've only used two objects, a and b , and got our interpretation: Rab and Rba are true.
- Is that allowed for the purpose of finding an open branch?
- Yes! Since it shows how we can make all formulae in the branch true ("true" according to their labels).

<i>inv</i>	$\mathbf{T}: \exists x Fx$
<hr style="width: 80%; margin: auto;"/>	
$\mathbf{T}: Fa \mid \mathbf{T}: Fb \mid \dots \mid \mathbf{T}: Fn$	
<i>for all a, b, ... in the branch or n new to the branch</i>	

Showing Invalidity (Example cont'd)

(n)	T:	$\forall x \exists y Rxy$	✓ ^{a,b}	from (k < n)
(n+1)	T:	$\exists y Ray$	✓	from (n)
(n+2)	T:	Rab		from (n+1)
(n+3)	T:	$\exists y Rby$	✓	from (n)
(n+4)	T:	Rba		from (n+3)

$X, \mathbf{T}: \forall x A$
<hr/>
$X, \mathbf{T}: \forall x A, \mathbf{T}: A_x^a$
<i>for a in X or A</i>

- Note that we would not even have needed b !
- We could also have constructed a model with just a , ending in line (n+2) with Raa

<i>inv</i>	T:	$\exists x Fx$							
<hr/>									
T:	Fa		T:	Fb		...		T:	Fn
<i>for all a, b, ... in the branch or n new to the branch</i>									

Simplifying a *true* \exists Quantifier (For Open Branches)

We can also use this *additional* rule:

<i>inv</i>	$X, \mathbf{T}: \exists x Fx$	<i>for all a, b, \dots in the branch or n new to the branch</i>
$X, \mathbf{T}: Fa \mid X, \mathbf{T}: Fb \mid \dots \mid X, \mathbf{T}: Fn$		

Why can you *replace* the original rule by this?

- For showing invalidity you just need to find any open branch, so having “too many branches” does not matter!
- For showing validity, you must show a contradiction in every single branch!
 - So adding these additional branches a, b, \dots , would only make your life harder, but can't cause a wrong outcome since the original branch (with the n) still remains in the rule!
 - But of course you would not add any of these new branches if you think the sequent is valid.

Exploiting the new Rule

- Although the new rule is only *necessary* for *some* invalid sequents (namely in those cases where we otherwise would run into infinite cycles), we still *could* (but don't have to!) exploit it even if we did not have to.
- This, however, does not always work. Consider the first two invalid sequents:
 - Example 1: $\forall x(Fx \vee Gx) \vdash \forall x Fx \vee \forall x Gx$
 - Example 2: $\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$
- These sequents are actually only invalid if there are at least two objects, if there were only one, then they *do* hold!
- Thus, applying the new rule in these examples would fail: we would generate only contradictions and thus had to branch using a new constant!

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$
- (2) **T:** $\exists x Gx$
- (3) **F:** $\exists x (Fx \wedge Gx)$

inv	T: $\exists x Fx$
T: Fa T: Fb ... T: Fn	
<i>for all a, b, ... in the branch or n new to the branch</i>	

F: $\exists x Fx$
F: $Fa, \mathbf{F}: Fb, \dots$
<i>for all a, b, ... in the branch – present and future!</i>

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$$

(1) **T:** $\exists x Fx$ ✓

(2) **T:** $\exists x Gx$

(3) **F:** $\exists x (Fx \wedge Gx)$

(4) **T:** Fa

from (1)

inv	T: $\exists x Fx$
T: Fa T: Fb ... T: Fn	
<i>for all a, b, ... in the branch or n new to the branch</i>	

F: $\exists x Fx$
F: $Fa, \mathbf{F: Fb}, \dots$
<i>for all a, b, ... in the branch – present and future!</i>

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$ ✓
 (2) **T:** $\exists x Gx$
 (3) **F:** $\exists x (Fx \wedge Gx)$ ✓^a
 (4) **T:** Fa from (1)
 (5) **F:** $Fa \wedge Ga$ from (3)

inv	T: $\exists x Fx$
T: Fa T: Fb ... T: Fn	
<i>for all a, b, ... in the branch or n new to the branch</i>	

F: $\exists x Fx$
F: $Fa, \mathbf{F}: Fb, \dots$
<i>for all a, b, ... in the branch – present and future!</i>

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$$

(1) **T:** $\exists x Fx$ ✓

(2) **T:** $\exists x Gx$

(3) **F:** $\exists x (Fx \wedge Gx)$ ✓^a

(4) **T:** Fa from (1)

(5) **F:** $Fa \wedge Ga$ ✓ from (3)

(6) **F:** Fa ↯ from (5) (6) **F:** Ga from (5)

inv

T: $\exists x Fx$

T: Fa | **T:** Fb | ... | **T:** Fn

*for all a, b, ... in the branch
or n new to the branch*

F: $\exists x Fx$

F: $Fa, \mathbf{F: Fb}, \dots$

*for all a, b, ...
in the branch –
present and future!*

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$$

(1) **T:** $\exists x Fx$ ✓

(2) **T:** $\exists x Gx$ ✓

(3) **F:** $\exists x (Fx \wedge Gx)$ ✓^a

(4) **T:** Fa from (1)

(5) **F:** $Fa \wedge Ga$ ✓ from (3)

(6) **F:** Fa ⚡ from (5) (6) **F:** Ga from (5)

(7) **T:** Ga ⚡ from (2)

(7) **T:** Gb from (2)

inv

T: $\exists x Fx$

T: Fa | **T:** Fb | ... | **T:** Fn

*for all a, b, ... in the branch
or n new to the branch*

F: $\exists x Fx$

F: $Fa, \mathbf{F: Fb}, \dots$

*for all a, b, ...
in the branch –
present and future!*

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$$

- (1) **T:** $\exists x Fx$ ✓
 (2) **T:** $\exists x Gx$ ✓
 (3) **F:** $\exists x (Fx \wedge Gx)$ ✓^{a,b}

(4) **T:** Fa from (1)

(5) **F:** $Fa \wedge Ga$ ✓ from (3)

(6) **F:** Fa ⚡ from (5) (6) **F:** Ga from (5)

(7) **T:** Ga ⚡ from (2)

(7) **T:** Gb from (2)

(8) **F:** $Fb \wedge Gb$ from (3)

inv**T:** $\exists x Fx$ **T:** Fa | **T:** Fb | ... | **T:** Fn

*for all a, b, ... in the branch
or n new to the branch*

F: $\exists x Fx$ **F:** $Fa, \mathbf{F}: Fb, \dots$

*for all a, b, ...
in the branch –
present and future!*

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$$

(1) **T:** $\exists x Fx$ ✓

(2) **T:** $\exists x Gx$ ✓

(3) **F:** $\exists x (Fx \wedge Gx)$ ✓^{*a,b*}

(4) **T:** Fa from (1)

(5) **F:** $Fa \wedge Ga$ ✓ from (3)

(6) **F:** Fa ⚡ from (5) (6) **F:** Ga from (5)

(7) **T:** Ga ⚡ from (2)

(7) **T:** Gb from (2)

(8) **F:** $Fb \wedge Gb$ ✓ from (3)

(9) **F:** Fb **open!** from (8) (9) **F:** Gb ⚡ from (8)

inv**T:** $\exists x Fx$ **T:** Fa | **T:** Fb | ... | **T:** Fn

*for all a, b, ... in the branch
or n new to the branch*

F: $\exists x Fx$ **F:** $Fa, \mathbf{F: Fb}, \dots$

*for all a, b, ...
in the branch –
present and future!*

Example 2 (Interpretation, Pun Intended)

- What did the previous example show us?
- It showed the importance of keeping the “original branch” of the original rule, as we might require it even for proving invalidity!
- I.e., in that example, branching over an existing constant (the left line (7)) wasn't successful!
- So like the previous proof with the 'standard rule' in example 2, we were able to obtain an interpretation showing that the sequent is invalid:
 - a is a footballer (line (4)) but not a goat (line (6))
 - b is is goat (line (7)) but not a footballer (line (9))
 - So $\exists x Fx, \exists x Gx \vdash \exists x (Fx \wedge Gx)$ is invalid.

Existential and Universal Quantifier for Invalid Sequents

<i>inv</i>	T: $\exists x Fx$
T: Fa T: Fb ... T: Fn	
<i>for all a, b, ... in the branch or n new to the branch</i>	

<i>inv</i>	F: $\forall x Fx$
F: Fa F: Fb ... F: Fn	
<i>for all a, b, ... in the branch or n new to the branch</i>	

≡

≡

<i>inv</i>	X, T: $\exists x A$
X, T: A_x^a X, T: A_x^b ... X, T: A_x^n	
<i>for any/all a, b, ... in X or A, or n not in X or A</i>	

<i>inv</i>	X, F: $\forall x A$
X, F: A_x^a X, F: A_x^b ... X, F: A_x^n	
<i>for any/all a, b, ... in X or A, or n not in X or A</i>	

On the right we see the dual to true existential.

Summary

Content of this Lecture

- Properties of Logics and Proof Systems
- Semantic Tableau for Predicate Logics
 - We added several additional rules, but kept using the old ones.
 - We still can prove validity and invalidity.
 - Invalidity cannot *a/ways* be proved!

Content of this Lecture

- Properties of Logics and Proof Systems
- Semantic Tableau for Predicate Logics
 - We added several additional rules, but kept using the old ones.
 - We still can prove validity and invalidity.
 - Invalidity cannot *a/ways* be proved!
- We added two alternative rules just for the purpose of finding an open branch that allow branching over existing constants.

Content of this Lecture

- Properties of Logics and Proof Systems
- Semantic Tableau for Predicate Logics
 - We added several additional rules, but kept using the old ones.
 - We still can prove validity and invalidity.
 - Invalidity cannot *a/ways* be proved!
- We added two alternative rules just for the purpose of finding an open branch that allow branching over existing constants.
- This week covered the following sections in the Logic Notes:
 - 5: More about first order logic
 - ▶ Quantifiers in semantic tableaux