Logic (PHIL 2080, COMP 2620, COMP 6262)
Chapter: First-Order Logic
— Properties of Proof Systems and Semantic Tableaux

Pascal Bercher

AI Group
School of Computing
College of Engineering and Computer Science
the Australian National University

pre-recorded for: 28 & 29 March 2022

Australian
National
University

Introduction

Recap: Predicate Logics

- We now (since week 5) know Predicate Logics as a means to express properties of and relationships between objects.
- For example:
  - If everyone plays football, and everyone is a goat, then everyone is a football-playing goat
  - $\forall x \, Fx, \forall x \, Gx \vdash \forall x \, (Fx \wedge Gx)$
- We know how to prove sequents involving Predicate Logic using Natural Deduction.
  - We “only” needed additional elimination and introduction rules for the exists ($\exists$) and universal ($\forall$) quantifiers.
  - Other than that we just re-used the rules for Propositional Logic.

Recap: Semantic Tableau

- Today, we cover Semantic Tableau for *Predicate Logic*.
- But first a recap on Semantic Tableau for *Propositional Logic*!
- If we want to prove $X \vdash A$ (with $X = \{A_1, \ldots, A_n\}$), then, we:
  - Label each assumption $A_1, \ldots, A_n$ as being *true* (**T**),
  - Label A as being *false* (**F**),
  - Simplify each formula (according to the connectives corresponding to truth tables) thus eventually obtaining:
    1. a contradiction in all the branches, or
    2. ≥ 1 open branch (i.e., none of its formulae can be simplified further and there’s no contradiction).

  In case 1 the sequent is *valid*.
  In case 2 the sequent is *invalid*, and we can construct an interpretation that makes all formulae in X true, but A false (which is a witness for invalidity).

Recap: Example for a Valid Sequent

$p \to q, r \to s \vdash (p \vee r) \to (q \vee s)$

(1) **T:** $p \to q$ ✓

(2) **T:** $r \to s$ ✓

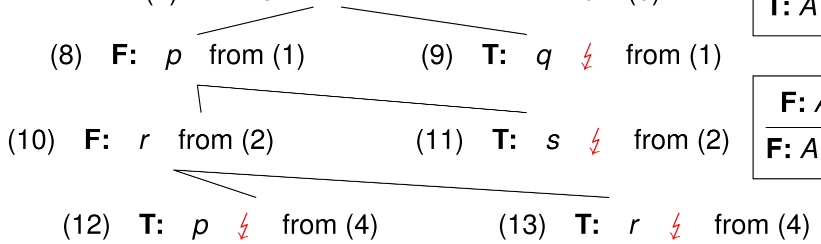
(3) **F:** $(p \vee r) \to (q \vee s)$ ✓

(4) **T:** $p \vee r$ ✓ from (3)

(5) **F:** $q \vee s$ ✓ from (3)

(6) **F:** q from (5)

(7) **F:** s from (5)



(8) **F:** p from (1) (9) **T:** q $\lightning$ from (1)

(10) **F:** r from (2) (11) **T:** s $\lightning$ from (2)

(12) **T:** p $\lightning$ from (4) (13) **T:** r $\lightning$ from (4)

T: $A \to B$
F: A | **T:** B

F: $A \to B$
T: A , **F:** B

T: $A \vee B$
T: A | **T:** B

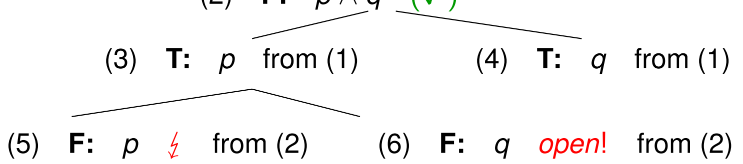
F: $A \vee B$
F: A , **F:** B

Recap: Example for an Invalid Sequent

$p \vee q \vdash p \wedge q$

(1) **T:** $p \vee q$ ✓

(2) **F:** $p \wedge q$ (✓)



(3) **T:** p from (1) (4) **T:** q from (1)

(5) **F:** p $\lightning$ from (2) (6) **F:** q *open*! from (2)

T: $A \vee B$
T: A | **T:** B

F: $A \wedge B$
F: A | **F:** B

- We've found an open branch, so we are allowed to stop the proof!
- The interpretation $I_1(p) = \top$ and $I_1(q) = \bot$ proves that the assumption $p \vee q$ can be made true while the formula $p \wedge q$ is false, thus invalidating the sequent.
- But we could also expand the right branch (line (4) due to line (2)) to obtain the second interpretation $I_2(p) = \bot$ and $I_2(q) = \top$. (Not required unless we explicitly ask you to.)

Today: Properties of Logics and Proof Systems

Properties of Logics:

- What does it mean to *decide* validity?
- Is that always possible for sequents in Propositional Logic? What about Predicate Logic?

Properties of Proof Systems:

- Are all proofs correct? (Soundness)
- Can we always prove validity? (Completeness)

Today: Semantic Tableau for Predicate Logic

- We still use the same rules as we had in the propositional case.
- But now we introduce four additional rules, namely for:
  - $\exists$-formulae which are labeled *true*
  - . . . *false*
  - $\forall$-formulae which are labeled *true*
  - . . . *false*
- We also introduce additional rules for the special case that we want to prove invalidity:
  - $\exists$-formulae which are labeled *true*
  - $\forall$-formulae . . . *false*
  - They will (sometimes) help to prove (more) sequents as invalid!

Properties

Recap on our Symbols and their Meanings

We differentiate between *validity* and *provability*:

- $X \models A$ (A follows logically from X)
  - → Every interpretation that makes X true also makes A true.
- $X \vdash_{ND} A$ ($X \vdash A$ can be proved via *Natural Deduction*)
  - → A can be *derived* from X. (Syntax manipulation.)
- $X \vdash_{ST} A$ ($X \vdash A$ can be proved via *Semantic Tableau*)
  - → We can't find an interpretation that makes X true but not A. (Exploits validity definition.)
- There are many more proof systems!

Syntax vs. Semantics

- So, what's the relation between $X \models A$ and $X \vdash A$?
- A desirable situation would be $X \models A$ iff $X \vdash A$.
- Our proof systems could do *anything*! So what *could* happen?
- Let $\mathcal{X}$ be *some* proof system (like, e.g., ND).
  1. $X \vdash_{\mathcal{X}} A$, but not $X \models A$
     - → The proof system is wrong! (I.e., not sound.)
  2. $X \models A$, but not $X \vdash_{\mathcal{X}} A$
     - → The proof system is incomplete! (I.e., not complete.)
- What we want:

Soundness	Every provable sequent is valid. (Cf. above's 1)
Completeness	Every valid sequent is provable. (Cf. above's 2)

Decision Procedure

- Let $\mathcal{X}$ be some proof system that's sound and complete.
- So, can we also *decide* validity of each sequent with $\mathcal{X}$?
- I.e., we want to know whether $X \models A$ holds, by using $\mathcal{X}$. Can we find out?
- Again, $\mathcal{X}$ is sound and complete, so we can check validity, right?
- No, not necessarily! Both just mention validity, not invalidity!
- We only know: $X \models A$ iff $X \vdash A$
- But we don't necessily know wheter $X \models A$ holds since a sequent could also be invalid! (In that case maybe the proof system just keeps running... So we don't get $X \vdash A$, but we also don't get an output saying "$X \models A$ is false")

Properties of Logics and proof systems

Decidability of Logics:

- *Decidability of a Logic* means determining for an arbitrary sequent whether it's valid or not.
- Propositional Logic: Yes, decidable.
- Predicate Logic: No, undecidable. No such algorithm can exist.

Soundness and Completeness of Proof Systems:

- Natural Deduction:
  - → Sound and complete for Propositional and Predicate Logic
- Semantic Tableau:
  - → Also Sound and complete for Propositional and Predicate Logic

---

Semantic Tableau Rules

Simplifying a *true* ∃ Quantifier (Intuition)

$$\frac{\textbf{T: } \exists x\ Fx}{\textbf{T: } Fa} \quad \text{provided } a \text{ is new to the branch}$$

- Why does *a* need to be new?
- Think of the triangle ABC! If *a* would exist already in the branch it would not be general (e.g., we could “accidentally” assume that our triangle is rectangular).

Simplifying a *false* ∀ Quantifier (Intuition)

$$\frac{\textbf{F: } \forall x\ Fx}{\textbf{F: } Fa} \quad \text{provided } a \text{ is new to the branch}$$

- This corresponds to the true existential quantifier!
- Recall $\neg\forall x\ Fx \equiv \exists x\ \neg Fx$

Rules For *true* ∃ and *false* ∀, formally

$$\frac{\textbf{T: } \exists x\ Fx}{\textbf{T: } Fa}$$

if a is new to the branch

$$\equiv$$

$$\frac{X, \textbf{T: } \exists x\ A}{X, \textbf{T: } A^a_x}$$

for a not in X or A

- The X represents all other lines we have in that branch.
- This notation shows nicely why we use checkmarks:
  - They show which lines have been processed.
  - Here it's those lines which don't appear below the line anymore! Note that **T:** $\exists x\ A$ doesn't appear anymore, meaning that the rule is "processed" already.

Rules For *true* ∃ and *false* ∀, formally

$$\frac{\textbf{T: } \exists x\ Fx}{\textbf{T: } Fa}$$

if a is new to the branch

$$\equiv$$

$$\frac{X, \textbf{T: } \exists x\ A}{X, \textbf{T: } A^a_x}$$

for a not in X or A

$$\frac{\textbf{F: } \forall x\ Fx}{\textbf{F: } Fa}$$

if a is new to the branch

$$\equiv$$

$$\frac{X, \textbf{F: } \forall x\ A}{X, \textbf{F: } A^a_x}$$

for a not in X or A

Simplifying a *true* ∀ Quantifier (Intuition)

$$\frac{\textbf{T: } \forall x\ Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all $a, b, \ldots$ in the branch (present and future!)

- This rule will continue being available for new constants/terms produced later on. (Then we have to apply the rule again!)
- If we already obtained a contradiction, we are clearly done. But if we want to show that a branch is open we need to have applied this rule to *all* constants! (I.e., also those that get created after we already applied the rule to all constants that existed until then.)

Simplifying a *false* ∃ Quantifier (Intuition)

$$\frac{\textbf{F: } \exists x\ Fx}{\textbf{F: } Fa, \textbf{F: } Fb, \ldots}$$

for all $a, b, \ldots$ in the branch (present and future!)

- Again, this rule will never be finished! If a new constant/term gets introduced we need to apply the rule again!
- Recall from last week that $\neg\exists x\ Fx \equiv \forall x\ \neg Fx$

Rules for *true* ∀ and *false* ∃, formally

$$\frac{\mathbf{T}\!: \forall x\ Fx}{\mathbf{T}\!: Fa, \mathbf{T}\!: Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\frac{\mathbf{F}\!: \exists x\ Fx}{\mathbf{F}\!: Fa, \mathbf{F}\!: Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\equiv \qquad \equiv$$

$$\frac{X, \mathbf{T}\!: \forall x\ A}{X, \mathbf{T}\!: \forall x\ A, \mathbf{T}\!: A^a_x}$$

for a in X or A

$$\frac{X, \mathbf{F}\!: \exists x\ A}{X, \mathbf{F}\!: \exists x\ A, \mathbf{F}\!: A^a_x}$$

for a in X or A

Special case for false Existential and true Universal

- Recall the rules for false existentials and true universals:

$$\frac{\mathbf{F}\!: \exists x\ Fx}{\mathbf{F}\!: Fa, \mathbf{F}\!: Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

$$\frac{\mathbf{T}\!: \forall x\ Fx}{\mathbf{T}\!: Fa, \mathbf{T}\!: Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

- They state that you only “use” constants which are already there.
- Sometimes, however, there one no such constants! Then, you are also allowed to create a new one.
- Do you need an example? Create one!

Examples

Quick Note

In the following examples we provide the rules in their non-general form (that uses an example!) – this just done for didactic reasons as this notation is easier to grasp.

Example 1

$\forall x(Fx \lor Gx) \vdash^? \forall x\, Fx \lor \forall x\, Gx$

(1) **T:** $\forall x(Fx \lor Gx)$
(2) **F:** $\forall x\, Fx \lor \forall x\, Gx$ ✓
(3) **F:** $\forall x\, Fx$ from (2)
(4) **F:** $\forall x\, Gx$ from (2)

$$\frac{\textbf{T: } \forall x\, Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all a, b, . . . in the branch – present and future!

$$\frac{\textbf{F: } \forall x\, Fx}{\textbf{F: } Fa}$$

if a is new to the branch

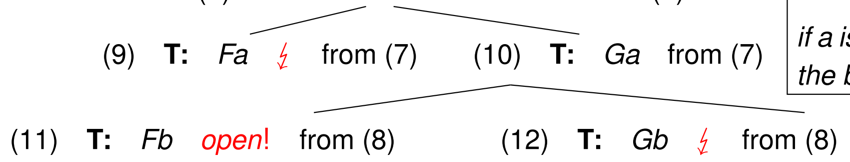
Trick question! :) (In the live lecture...)

Note that we did not apply the rule for false universal quantifier here because the formula is actually a false *disjunction*, not a false universally quantified formula.

Example 1

$\forall x(Fx \lor Gx) \vdash^? \forall x\, Fx \lor \forall x\, Gx$

(1) **T:** $\forall x(Fx \lor Gx)$ $✓^{a,b}$
(2) **F:** $\forall x\, Fx \lor \forall x\, Gx$ ✓
(3) **F:** $\forall x\, Fx$ ✓ from (2)
(4) **F:** $\forall x\, Gx$ ✓ from (2)
(5) **F:** Fa from (3)
(6) **F:** Gb from (4)
(7) **T:** $Fa \lor Ga$ ✓ from (1)
(8) **T:** $Fb \lor Gb$ ✓ from (1)



(9) **T:** Fa ↯ from (7)
(10) **T:** Ga from (7)
(11) **T:** Fb *open*! from (8)
(12) **T:** Gb ↯ from (8)

Extracted interpretation: see next slide.

$$\frac{\textbf{T: } \forall x\, Fx}{\textbf{T: } Fa, \textbf{T: } Fb, \ldots}$$

for all a, b, . . . in the branch – present and future!

$$\frac{\textbf{F: } \forall x\, Fx}{\textbf{F: } Fa}$$

if a is new to the branch

Example 1 (cont'd)

So? Is $\forall x(Fx \lor Gx) \vdash \forall x\, Fx \lor \forall x\, Gx$ valid?

- Let's see... Not all branches are contradictory.
- Thus, there is an open branch:

(5) **F:** Fa from (3)
(6) **F:** Gb from (4)
(10) **T:** Ga from (7)
(11) **T:** Fb from (8)

- We can, as usual, extract an *interpretation I* that answers for which objects F and G is true:
  - Informally: $I(Fa) = \bot$ and $I(Fb) = \top$
  - Informally: $I(Ga) = \top$ and $I(Gb) = \bot$
  - Thus, showing that there is an interpretation that makes the assumption true, but the formula false!
  - → So the sequent is invalid!

The formal definition will be provided in week 7

Example 1 (cont'd, once more!)

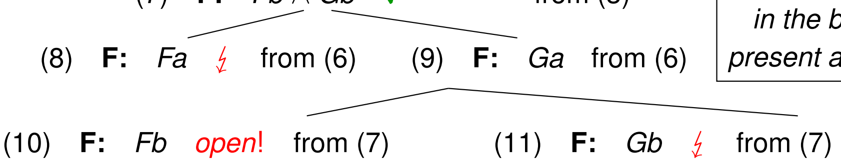
What's the *intuitive* way of interpreting the invalidity of $\forall x(Fx \lor Gx) \vdash \forall x\, Fx \lor \forall x\, Gx$?

- There's a "world" in which the the sequent doesn't hold!
- The sequent would be valid if in all "worlds" it holds:
  - If all "individuals" are footballers or goats,
  - then all individuals are footballers or all individuals are goats.
- It certainly *can* be true!
  - If there's just one individual! Let's call it Pascal and assume he/it is a footballer, a goat, or both. Then the sequent is valid!
  - Let *a* and *b* be footballers, none of them is a goat. Still valid!
- But in our counter-example world it's not true!
  - *a* is a goat, *b* is a footballer. So everything is a footballer or a goat.
  - But it's not true that *a* and *b* are footballers or *a* and *b* are goats!

Example 2

$\exists x\ Fx, \exists x\ Gx \vdash^{?} \exists x\ (Fx \land Gx)$

(1) **T:** $\exists x\ Fx$ ✓
(2) **T:** $\exists x\ Gx$ ✓
(3) **F:** $\exists x\ (Fx \land Gx)$ $\checkmark^{a,b}$
(4) **T:** Fa from (1)
(5) **T:** Gb from (2)
(6) **F:** $Fa \land Ga$ ✓ from (3)
(7) **F:** $Fb \land Gb$ ✓ from (3)


(8) **F:** Fa ↯ from (6)
(9) **F:** Ga from (6)
(10) **F:** Fb open! from (7)
(11) **F:** Gb ↯ from (7)

T: $\exists x\ Fx$
T: Fa
if a is new to the branch

F: $\exists x\ Fx$
F: Fa, **F:** $Fb, \ldots$
for all $a, b, \ldots$ in the branch – present and future!

Extracted interpretation: see next slide.

Example 2 (cont'd)

So? Is $\exists x\ Fx, \exists x\ Gx \vdash^{?} \exists x\ (Fx \land Gx)$ valid?

- Let's see... Not all branches are contradictory.
- Thus, there is an open branch:

  (4) **T:** Fa from (1)
  (5) **T:** Gb from (2)
  (9) **F:** Ga from (6)
  (10) **F:** Fb from (7)

- Again we can design an interpretation that answers for which objects F and G become true:
  - F is true for exactly a
  - G is true for exactly b
  - Thus, showing that there is an interpretation that makes the assumption true, but the formula false!
  - → So the sequent is invalid!

Example by de'Morgan

Intended to show how Predicate Logic goes beyond Propositional Logic:

- All horses are animals.
- Therefore, any horse's head is an animal head!

We formalize this in terms of Predicate Logic.

- Instead of:
  "any horse's head is an animal head"
- We formalize that as: "each part of a horse is part of an animal"
  $\forall x(\exists y(Hy \land Pxy) \rightarrow \exists y(Ay \land Pxy))$

Thus we get:

$\forall x\ Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \land Pxy) \rightarrow \exists y(Ay \land Pxy))$

Example 3

$\forall x\ Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \land Pxy) \rightarrow \exists y(Ay \land Pxy))$

(1) **T:** $\forall x\ Hx \rightarrow Ax$
(2) **F:** $\forall x(\exists y(Hy \land Pxy) \rightarrow \exists y(Ay \land Pxy))$ ✓
(3) **F:** $\exists y(Hy \land Pay) \rightarrow \exists y(Ay \land Pay)$ ✓ from (2)
(4) **T:** $\exists y(Hy \land Pay)$ ✓ from (3)
(5) **F:** $\exists y(Ay \land Pay)$ from (3)
(6) **T:** $Hb \land Pab$ ✓ from (4)
(7) **T:** Hb from (6)
(8) **T:** Pab from (6)

T: $\forall x\ Fx$
T: Fa, **T:** $Fb, \ldots$
for all $a, b, \ldots$ in the branch – present and future!

F: $\forall x\ Fx$
F: Fa
if a is new to the branch

F: $\exists x\ Fx$
F: Fa, **F:** $Fb, \ldots$
for all $a, b, \ldots$ in the branch – present and future!

T: $\exists x\ Fx$
T: Fa
if a is new to the branch

Example 3

$\forall x\, Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$

(1)	**T:**	$\forall x\, Hx \rightarrow Ax$ $\checkmark^b$	
(2)	**F:**	$\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓	
(3)	**F:**	$\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓	from (2)
(4)	**T:**	$\exists y(Hy \wedge Pay)$ ✓	from (3)
(5)	**F:**	$\exists y(Ay \wedge Pay)$ $\checkmark^b$	from (3)
(6)	**T:**	$Hb \wedge Pab$ ✓	from (4)
(7)	**T:**	Hb	from (6)
(8)	**T:**	Pab	from (6)
(9)	**T:**	$Hb \rightarrow Ab$ ✓	from (1)

(10) **F:** Hb ↯ from (9)	(11) **T:** Ab from (9)
	(12) **F:** $Ab \wedge Pab$ ✓ from (5)

(13) **F:** Ab ↯ from (12)	(14) **F:** Pab ↯ from (12)

All branches are contradictory. Sequent is *valid*!

Example 3 (Again with a different Order)

$\forall x\, Hx \rightarrow Ax \vdash \forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$

(1)	**T:**	$\forall x\, Hx \rightarrow Ax$ $\checkmark^b$	
(2)	**F:**	$\forall x(\exists y(Hy \wedge Pxy) \rightarrow \exists y(Ay \wedge Pxy))$ ✓	
(3)	**F:**	$\exists y(Hy \wedge Pay) \rightarrow \exists y(Ay \wedge Pay)$ ✓	from (2)
(4)	**T:**	$\exists y(Hy \wedge Pay)$ ✓	from (3)
(5)	**F:**	$\exists y(Ay \wedge Pay)$ $\checkmark^b$	from (3)
(6)	**T:**	$Hb \wedge Pab$ ✓	from (4)
(7)	**T:**	Hb	from (6)
(8)	**T:**	Pab	from (6)
(9)	**F:**	$Ab \wedge Pab$ ✓	from (5)

(10) **F:** Ab from (9)	(11) **F:** Pab ↯ from (9)
(12) **T:** $Hb \rightarrow Ab$ ✓ from (1)	

(13) **F:** Hb ↯ from (12)	(14) **T:** Ab ↯ from (12)

All branches are contradictory. Sequent is *valid*!

Invalid Sequents

Motivation: Sequent is invalid, so?

- You learned that there are some invalid sequents for which you can't find a proof that shows invalidity.
- (We were however still able to find invalidity proofs for *some* invalid sequents, e.g., examples 1 and 2.)
- In some cases however, we *could* prove invalidity if we had more clever rules – which are tailored for finding invalid proofs.
- Even with these rules, though, we still can't *always* prove invalidity. (Since Predicate Logic is undecidable.)

Motivation: Motivating Example

Assume we are deep within some branch:

(n)	**T:**	$\forall x \, \exists y \, Rxy$	$\checkmark^{a,b,c}$	from (k<n)
(n+1)	**T:**	$\exists y \, Ray$	✓	from (n)
(n+2)	**T:**	Rab		from (n+1)
(n+3)	**T:**	$\exists y \, Rby$	✓	from (n)
(n+4)	**T:**	Rbc		from (n+3)
(n+5)	**T:**	$\exists y \, Rcy$	✓	from (n)
(n+6)	**T:**	Rcd		from (n+5)

$$\frac{X, \textbf{T: } \forall x \, A}{X, \textbf{T: } \forall x \, A, \textbf{T: } A^a_x}$$

for a in X or A

$$\frac{X, \textbf{T: } \exists x \, A}{X, \textbf{T: } A^a_x}$$

for a not in X or A

- So we have an infinite branch!
- We will *never* be able to show that it is open.
- Is this what we want?
- I.e., *should* we be able to prove that the branch is open?

Motivation: Showing Invalidity

- Recall what an invalid sequent $X \vdash A$ means, and how we do it:
  - It means that we can find an interpretation (“special case”) where all premises (i.e., formulae in X) are true, but not the formula A. So the formula A is not a logical consequence.
  - We prove that by providing such a “special case” interpretation!
  - More precisely, we derive an interpretation that is consistent with all formulae in the current branch, thus showing that it’s not always contradictory!
- In order to achieve our goal, i.e., find an interpretation for such an infinite branch, we will have to alter one of our rules.

Rules & Example

Showing Invalidity (Example cont’d)

(n)	**T:**	$\forall x \, \exists y \, Rxy$	$\checkmark^{a,b}$	from (k<n)
(n+1)	**T:**	$\exists y \, Ray$	✓	from (n)
(n+2)	**T:**	Rab		from (n+1)
(n+3)	**T:**	$\exists y \, Rby$	✓	from (n)
(n+4)	**T:**	Rba		from (n+3)

$$\frac{X, \textbf{T: } \forall x \, A}{X, \textbf{T: } \forall x \, A, \textbf{T: } A^a_x}$$

for a in X or A

- Now we’ve only used two objects, *a* and *b*, and got our interpretation: *Rab* and *Rba* are true.
- Is that allowed for the purpose of finding an open branch?
- Yes! Since it shows how we can make all formulae in the branch true (“true” according to their labels).

inv

$$\frac{\textbf{T: } \exists x \, Fx}{\textbf{T: } Fa \mid \textbf{T: } Fb \mid \ldots \mid \textbf{T: } Fn}$$

for all a, b, . . . in the branch or n new to the branch

Showing Invalidity (Example cont'd)

(n) **T:** $\forall x\,\exists y\; Rxy$ ✓^{a,b} from (k<n)
(n+1) **T:** $\exists y\; Ray$ ✓ from (n)
(n+2) **T:** Rab from (n+1)
(n+3) **T:** $\exists y\; Rby$ ✓ from (n)
(n+4) **T:** Rba from (n+3)

$$\frac{X, \mathbf{T{:}}\ \forall x\ A}{X, \mathbf{T{:}}\ \forall x\ A, \mathbf{T{:}}\ A^a_x}$$

for a in X or A

- Note that we would not even have needed *b*!
- We could also have constructed a model with just *a*, ending in line (n+2) with *Raa*

$$\boxed{inv}\quad \frac{\mathbf{T{:}}\ \exists x\ Fx}{\mathbf{T{:}}\ Fa \mid \mathbf{T{:}}\ Fb \mid \ldots \mid \mathbf{T{:}}\ Fn}$$

for all a, b, . . . in the branch
or n new to the branch

Simplifying a *true* ∃ Quantifier (For Open Branches)

We can also use this *additional* rule:

$$\boxed{inv}\quad \frac{X, \mathbf{T{:}}\ \exists x\ Fx}{X, \mathbf{T{:}}\ Fa \mid X, \mathbf{T{:}}\ Fb \mid \cdots \mid X, \mathbf{T{:}}\ Fn}$$

for all a, b, . . . in the branch
or n new to the branch

Why can you *replace* the original rule by this?

- For showing invalidity you just need to find any open branch, so having "too many branches" does not matter!
- For showing validity, you must show a contradiction in every single branch!
  - So adding these additional branches *a*, *b*, . . . , would only make your life harder, but can't cause a wrong outcome since the original branch (with the *n*) still remains in the rule!
  - But of course you would not add any of these new branches if you think the sequent is valid.

Exploiting the new Rule

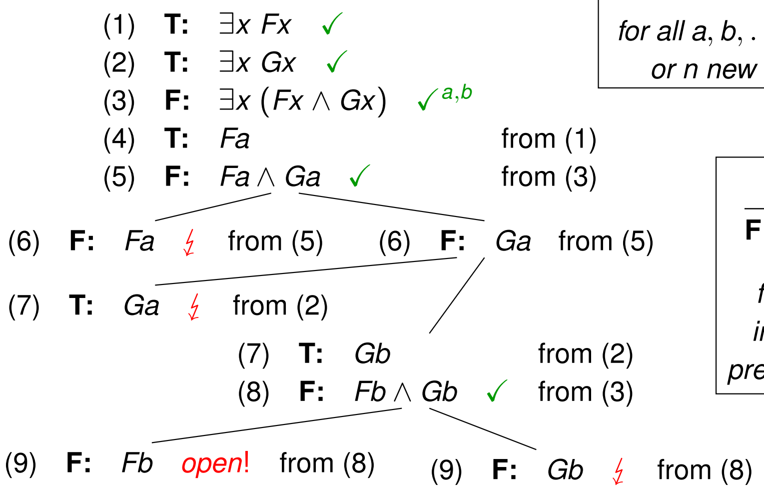
- Although the new rule is only *necessary* for *some* invalid sequents (namely in those cases where we otherwise would run into infinite cycles), we still *could* (but don't have to!) exploit it even if we did not have to.
- This, however, does not always work. Consider the first two invalid sequents:
  - Example 1: $\forall x(Fx \lor Gx) \vdash \forall x\ Fx \lor \forall x\ Gx$
  - Example 2: $\exists x\ Fx, \exists x\ Gx \vdash \exists x\ (Fx \land Gx)$
- These sequents are actually only invalid if there are at least two objects, if there were only one, then they *do* hold!
- Thus, applying the new rule in these examples would fail: we would generate only contradictions and thus had to branch using a new constant!

Example 2 (Failed Attempt Using (Shortcut-branches of) new Rule)

$$\exists x\ Fx, \exists x\ Gx \vdash \exists x\ (Fx \land Gx)$$

$$\boxed{inv}\quad \frac{\mathbf{T{:}}\ \exists x\ Fx}{\mathbf{T{:}}\ Fa \mid \mathbf{T{:}}\ Fb \mid \ldots \mid \mathbf{T{:}}\ Fn}$$

for all a, b, . . . in the branch
or n new to the branch



(1) **T:** $\exists x\ Fx$ ✓
(2) **T:** $\exists x\ Gx$ ✓
(3) **F:** $\exists x\ (Fx \land Gx)$ ✓^{a,b}
(4) **T:** Fa from (1)
(5) **F:** $Fa \land Ga$ ✓ from (3)
(6) **F:** Fa ↯ from (5) (6) **F:** Ga from (5)
(7) **T:** Ga ↯ from (2)
(7) **T:** Gb from (2)
(8) **F:** $Fb \land Gb$ ✓ from (3)
(9) **F:** Fb open! from (8) (9) **F:** Gb ↯ from (8)

$$\frac{\mathbf{F{:}}\ \exists x\ Fx}{\mathbf{F{:}}\ Fa, \mathbf{F{:}}\ Fb, \ldots}$$

for all a, b, . . .
in the branch –
present and future!

Example 2 (Interpretation, Pun Intended)

- What did the previous example show us?
- It showed the importance of keeping the "original branch" of the original rule, as we might require it even for proving invalidity!
- I.e., in that example, branching over an existing constant (the left line (7)) wasn't successful!
- So like the previous proof with the 'standard rule' in example 2, we were able to obtain an interpretation showing that the sequent is invalid:
  - *a* is a footballer (line (4)) but not a goat (line (6))
  - *b* is is goat (line (7)) but not a footballer (line (9))
  - So $\exists x\ Fx, \exists x\ Gx \vdash \exists x\ (Fx \wedge Gx)$ is invalid.

Existential and Universal Quantifier for Invalid Sequents

$$\boxed{inv}\quad \frac{\mathbf{T}\!: \exists x\ Fx}{\mathbf{T}\!: Fa \mid \mathbf{T}\!: Fb \mid \ldots \mid \mathbf{T}\!: Fn}$$

for all a, b, . . . in the branch
or n new to the branch

$$\boxed{inv}\quad \frac{\mathbf{F}\!: \forall x\ Fx}{\mathbf{F}\!: Fa \mid \mathbf{F}\!: Fb \mid \ldots \mid \mathbf{F}\!: Fn}$$

for all a, b, . . . in the branch
or n new to the branch

$$\equiv \qquad\qquad \equiv$$

$$\boxed{inv}\quad \frac{X, \mathbf{T}\!: \exists x\ A}{X, \mathbf{T}\!: A^a_x \mid X, \mathbf{T}\!: A^b_x \mid \cdots \mid X, \mathbf{T}\!: A^n_x}$$

for any/all a, b, . . . in X or A,
or n not in X or A

$$\boxed{inv}\quad \frac{X, \mathbf{F}\!: \forall x\ A}{X, \mathbf{F}\!: A^a_x \mid X, \mathbf{F}\!: A^b_x \mid \cdots \mid X, \mathbf{F}\!: A^n_x}$$

for any/all a, b, . . . in X or A,
or n not in X or A

On the right we see the dual to true existential.

Summary

Content of this Lecture

- Properties of Logics and Proof Systems
- Semantic Tableau for Predicate Logics
  - We added several additional rules, but kept using the old ones.
  - We still can prove validity and invalidity.
  - Invalidity cannot *always* be proved!
- We added two alternative rules just for the purpose of finding an open branch that allow branching over existing constants.
- This week covered the following sections in the Logic Notes:
  - 5: More about first order logic
    - Quantifiers in semantic tableaux