Algorithms (COMP3600/6466)
Data Structures: Heaps

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Introduction

Motivation

Recap that we want to do (at least) the following operations efficiently:

- access, i.e., search
- min/max
- insertion/deletion

Which runtime did we have for binary search trees?

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Recap that we want to do (at least) the following operations efficiently:

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Which runtime did we have for binary search trees?
$O(h)$, where h is the tree's height.

We now try to do better.

Overview

Existing operations for heaps:

- Heapify to ensure/establish heap properties
- Insertion
- ExtractMax (i.e., find and remove maximum)

All of these operations run in $O(log(n))$ (instead of $O(h)$).

Basics

Heap

A heap is a binary tree that satisfies the *heap property*.
I.e., it holds:

- A heap is a:
  - *complete binary tree*, i.e., a perfect binary tree where missing nodes might only be right-most leaves in the last level.
  - Def.: *perfect binary tree*: all interior nodes have two children, and and all leaves are at the same level.

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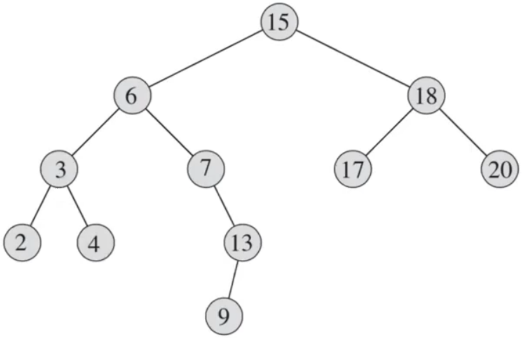
  - Def.: *perfect binary tree*: all interior nodes have two children, and and all leaves are at the same level.
- Same data management as for the binary search tree:
  - Each node contains a key.
  - Each node may have satellite data.
- Each parent node has a key greater than the keys of its children. This is a *Max-heap*. Min-heaps can be defined analogously. (We only consider Max-heaps.)

Examples

Is the following graph a heap?



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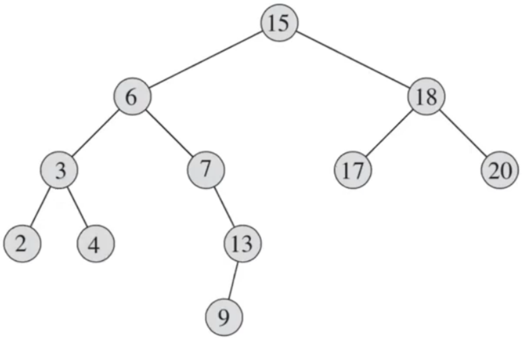

→ No, e.g., 15 and 18 are wrongly ordered (for a max heap).
And it's not complete.

Examples

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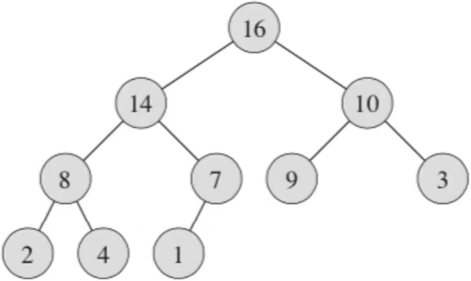

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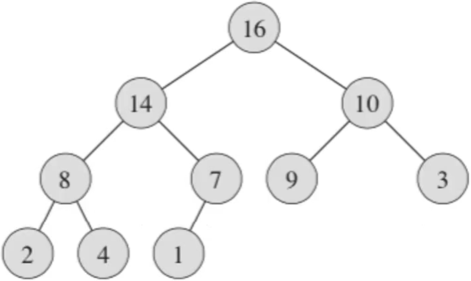
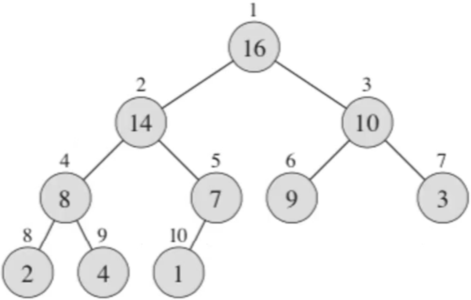


$\rightarrow$ Yes (a Max-heap)

Efficient Implementation of Heaps

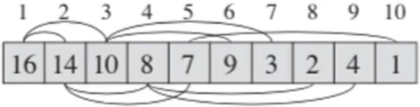
They can be stored as arrays:



(Because it's complete!)

$\textsc{Parent}(i) = \lfloor i/2 \rfloor$
$\textsc{Left}(i) = 2i$
$\textsc{Right}(i) = 2i + 1$
where i is the array position.

E.g.,
$\textsc{Parent}(3) = \lfloor 3/2 \rfloor = 1$
$\textsc{Left}(3) = 2 \cdot 3 = 6$
$\textsc{Right}(3) = 2 \cdot 3 + 1 = 7$



Heapify

Assumptions & Terminology

The **Heapify algorithm** (one call!) will be used (among others) to:

- Create a heap from an unsorted array, runs in $O(n)$
- sort an array runs in $O(n \cdot log(n))$

Assumptions & Terminology

The **Heapify algorithm** (one call!) will be used (among others) to:

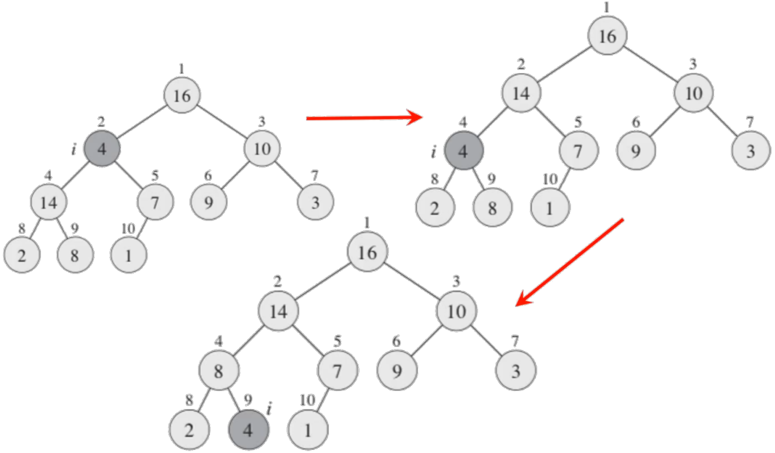
- Create a heap from an unsorted array, runs in $O(n)$
- sort an array runs in $O(n \cdot log(n))$

But the heapify algorithm itself is a single call, running in $O(log(n))$.
It assumes:

- We have a node at index i and,
- the heap property holds for both LEFT(i) and RIGHT(i),
- but $A[i]$ might be smaller than its children.

Example

To heapify a node (that's in the tree but violating the heap property) means to traverse the tree downwards (from it) re-ordering the respective branch by switching places with the maximum.



Algorithm

```
MAX-HEAPIFY(A, i)
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    largest = l
else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
    largest = r
if largest ≠ i
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest)
```

Note that this algorithm calls itself again on one of *i*'s children.

Runtime

The runtime is (rather obviously) in $O(log(n))$, why?

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The runtime is (rather obviously) in $O(log(n))$, why?

- Once heapify was called for a node x (taking constant time), it is called for only *one* of its children.
- How often can we invoke it again?
  $\rightarrow$ as often as there are children!

Since the height of a complete binary tree with n nodes is $log(n)$ we get runtime of $O(log(n))$.

Runtime (alternative proof)

We also obtain $O(log(n))$ by solving the following equation:

$$T(n) \leq T(\frac{2}{3}n) + c,$$

where T is the actual runtime of the problem (and n the number of nodes and c a constant).

That the equation only has a solution for $T(n) \in O(log(n))$ follows from the *Master theorem* (proved earlier by Ahad).

We thus only show why the equation itself holds.

Runtime (alternative proof, cont'd: why does $T(n) \leq T(\frac{2}{3}n) + c$ hold?

- We know that a call to i will perform constant (c) effort and then invoke the algorithm again for one of its children.
- So we can estimate the worst-case number of nodes that the larger sub tree may have:
  $n = 1+$ # nodes in left subtree $+$ # nodes in right subtree
- The left subtree is one level deeper than the right.

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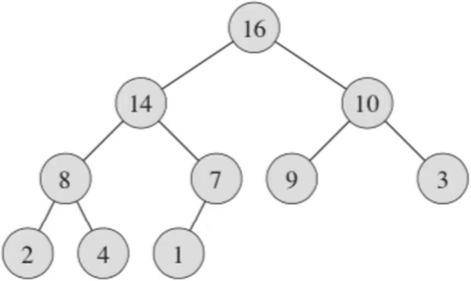
Insertion & Increase Key

Example

- Assume a given heap. We want to insert a key and establish the heap property again.
- Intuition: Insert it at the “next free” position and move it to an adequate position afterwards.



Name any number to insert!

Algorithm

```
MAX-HEAP-INSERT(A, key)
A.heap-size = A.heap-size + 1
A[A.heap-size] = -∞
HEAP-INCREASE-KEY(A, A.heap-size, key)
```

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

Runtime

Runtime of this code:

```
MAX-HEAP-INSERT(A, key)
1  A.heap-size = A.heap-size + 1
2  A[A.heap-size] = -∞
3  HEAP-INCREASE-KEY(A, A.heap-size, key)
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```
HEAP-INCREASE-KEY(A, i, key)
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3  A[i] = key
4  while i > 1 and A[PARENT(i)] < A[i]
5      exchange A[i] with A[PARENT(i)]
6      i = PARENT(i)
```

- In the worst case, lines 4–6 of are called until the root is reached.
- Therefore, the time complexity is $O(h) = O(log(n))$.

Build Heap

Build Heap – *not*

- We could build a heap of size n by inserting n times.
- However, that would lead to a runtime of $O(n \cdot log(n))$.
- We can do better!

Algorithm & Example

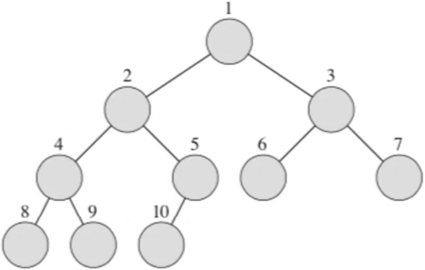
```
BUILD-MAX-HEAP(A)
    A.heap-size = A.length
    for i = ⌊A.length/2⌋ downto 1
        MAX-HEAPIFY(A, i)
```

Why do we start at the middle of the array and walk to the left?

Because Heapify assumes that LEFT(i) and RIGHT(i) satisfies heap properties! So we must work bottom-up!

Example:

A	4	1	3	2	16	9	10	14	8	7



Runtime

- Heapify (runtime $O(log(n))$) is called $\frac{n}{2}$ times, so it still appears as $O(n \cdot log(n))$.
- But we claimed we could to better, $O(n)$!
  What?! Were we wrong??

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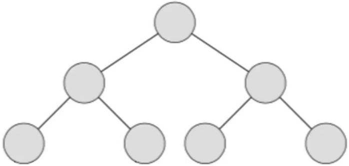
- But we claimed we could to better, $O(n)$!
  What?! Were we wrong??
- No! Not each call has runtime $O(log(n))$!
- Our analysis actually showed $O(log(h_i))$ for the height h_i of the "start node". But the height changes! And there are much more nodes on lower than on higher levels!
- As an intuition, recall that in a perfect binary tree, roughly 50% of all nodes are in the last layer, so half of our calls take constant time!

Runtime, cont'd

Let $T(n)$ be the actual runtime for a tree with n nodes.

$$T(n) \leq \sum_{h=0}^{\lfloor log(n) \rfloor} \lceil \frac{n}{2^{h+1}} \rceil O(h)$$



Number of nodes at $h = 0$ is $\lceil \frac{7}{2^{0+1}} \rceil = \lceil \frac{7}{2} \rceil = \lceil 3.5 \rceil = 4$ at the bottom, then $\lceil \frac{7}{2^{1+1}} \rceil = 2$ in the middle for $h = 1$, etc.

$\lceil \frac{n}{2^{h+1}} \rceil$ refers to the number of nodes per level (at "current height h").

Important: This height is relative to where we start Heapify, so $h = 0$ is the *leafs*, *not the root*!

Runtime, cont'd

Let $T(n)$ be the actual runtime for a tree with n nodes.

$$T(n) \leq \sum_{h=0}^{\lfloor log(n) \rfloor} \lceil \frac{n}{2^{h+1}} \rceil O(h) \leq \sum_{h=0}^{\lfloor log(n) \rfloor} \lceil \frac{n}{2^{h+1}} \rceil c \cdot h$$

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$$\leq c \cdot \sum_{h=0}^{\lfloor log(n) \rfloor} \frac{n}{2^h} h \leq c \cdot n \cdot \sum_{h=0}^{\lfloor log(n) \rfloor} \frac{h}{2^h}$$

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$$\leq c \cdot \sum_{h=0}^{\lfloor log(n) \rfloor} \frac{n}{2^h} h \leq c \cdot n \cdot \sum_{h=0}^{\lfloor log(n) \rfloor} \frac{h}{2^h} \leq c \cdot n \cdot \sum_{h=0}^{\infty} \frac{h}{2^h}$$

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$$\leq c \cdot \sum_{h=0}^{\lfloor log(n) \rfloor} \frac{n}{2^h} h \leq c \cdot n \cdot \sum_{h=0}^{\lfloor log(n) \rfloor} \frac{h}{2^h} \leq c \cdot n \cdot \sum_{h=0}^{\infty} \frac{h}{2^h} \leq c \cdot n \cdot 2$$

Thus we get $T(n) \in O(n)$.

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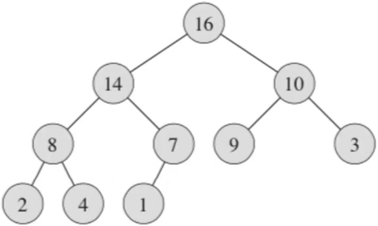
Important: This height is relative to where we start Heapify, so $h = 0$ is the *leafs*, *not the root*!

Extract Max

Algorithm, Example, and Runtime

```
HEAP-EXTRACT-MAX(A)
if A.heap-size < 1
    error "heap underflow"
max = A[1]
A[1] = A[A.heap-size]
A.heap-size = A.heap-size − 1
MAX-HEAPIFY(A, 1)
return max
```



The algorithm only needs to traverse a path of the tree once. Hence, the complexity is $O(log(n))$

Applications

Heap Sort

- To sort an array, create a heap, then extract all max values one by one.
- Complexity: $O(n \cdot log(n))$
- In practice, QuickSort runs faster than Heap Sort.
- But the worst-case of heap sort is better!

Priority Queue

A priority queue is a data structure that maintains a set S, where each element is associated with a key. It features the following operations:

- Insert(S, x): Inserts element x into the set S
- Maximum(S): Returns an element of S with the largest key
- ExtractMax(S): Removes and returns an element of S with the largest key
- IncreaseKey (S, x, k): Increase the key of x to k

Common application: Search, e.g., in Automated Planning.
Here we sort by minimum, e.g., for minimal $f(n) = g(n) + h(n)$ in A^*.

Summary

Summary

Today we covered **Heaps**.

Operations considered:

- Heapify $O(log(n))$
- Insertion $O(log(n))$
- Increase-Key $O(log(n))$
- Extract-Max $O(log(n))$
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Applications mentioned:

- Sorting arrays $O(n \cdot log(n))$
- Priority Queues