COMP3630 / COMP6363

week 11: **Alternating Time**

This Chapter is not coverd in HMU

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Content of this Chapter

- Alternating Turing Machines (ATMs)
- The complexity class **AP**
- **AP** vs. **PSPACE**

The Geography Game

Rules of Geography given a designated starting city (e.g. London)

1. Player 1 names a city that begins with the last letter of the designated city (e.g. Newcastle) and makes this the designated city.
2. Player 2 names a city that begins with the last letter of the city named by player 2 (e.g. Edinburgh) and makes this the designated city, continue with rule 1.

Winning Conditions.

- The game is lost by the player that cannot name a city and won by the other player.

Question.

Does Player 1 have a winning strategy (i.e. can always win irrespective of the moves of player one)?

(In "reality" we have partial knowledge but a hypothesis about what the other player knows (epistemic reasoning). Here we assume full knowledge (full observability.))

The Proof Game

Background.

- A formula A is provable if there is a proof rule with conclusion A, such that all its premisses are provable (e.g. $\frac{B \to A \qquad B}{A}$)

Rules of the Proof Game for a given designated formula A_0:

1. Player 1 chooses a proof rule $\frac{A_1 \quad \dots \quad A_n}{A_0}$ whose conclusion is the designated formula.
2. Player 2 chooses a premise A_i of the rule, and makes A_i the designated formula, continue with rule 1.

Winning conditions.

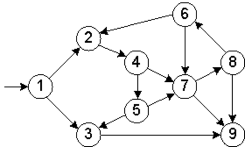
- the player who cannot move loses the game
- infinite plays are lost by Player 1

Question.

Does Player 1 have a winning strategy (i.e. can always win irrespective of the moves of player one) so that A is provable?

The Generalised Geography Game

From Geography to Generalised Geography: Replace cities with directed graph:



Rules.

- One node is always a designated node;
- Player 1 chooses a successor of the designated node which is the new designated node.
- Player 2 chooses a successor of the designated node which is the new designated node, continue with rule 1.

Winning Conditions.

- who cannot move, loses
- Player 2 wins infinite plays

Question.

What is the complexity that – given graph G with designated initial node – of determining whether Player 1 has a winning strategy?

Problem Reductions between these Games

From Geography to Generalised Geography.

Construct a graph where:

- the nodes are the names of cities
- there is an edge between city 1 and city 2 if the name of city 2 begins with the last letter of the name of city 1

From Proof to Generalised Geography.

Construct a graph where:

- nodes are either formulae, or proof rules
- there is an edge between a formula node A and a proof rule node $\frac{A_1 \quad \dots \quad A_n}{A_0}$ if $A = A_0$
- there is an edge between a proof rule node $\frac{A_1 \quad \dots \quad A_n}{A_0}$ and a formula node A if $A = A_i$, for some $1 \leq i \leq n$.

Winning Strategies (for any 2-Player Game!)

For Player 1 to win from starting node n:

- there exists a move such that for all moves of player 2 to node n' ...
- Player 1 has a winning strategy from node n'

Pattern for winning strategy:

- existential choice for player 1
- universal choice for player 2

Recap: Non-deterministic Machines

Complexity Class NP.

Have non-deterministic machine

- where every run takes at most polynomially many steps
- there exists an accepting sequence of IDs

Complexity Class co-NP.

Have non-deterministic machine

- where every run takes at most polynomially many steps
- every sequence of IDs is accepting

Alternating Turing machines combine existential and universal runs

Alternating Turing Machines

Definition. An alternating Turing machine is a non-deterministic Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ where additionally $Q = Q_e \cup Q_u$ is partitioned into a set of Q_e of existential states and Q_u of universal states.

Instantaneous Descriptions (IDs)

- are defined as for non-deterministic machines, and contain tape content, head position, and state
- the transition relation $I \vdash J$ between IDs is defined as for non-deterministic machines
- an ID is existential if the state is existential, and universal, if the state is universal.

Q. What about acceptance . . . ?

Acceptance Conditions

Informally. An ATM M accepts string w if there is a finite tree whose nodes are IDs and

- the root node is the initial ID (w on tape, state q_0),
- every existential ID E has (exactly) one child J in the tree with $E \vdash J$
- every universal ID U has all IDs J with $U \vdash J$ as children, and
- all leaf nodes are universal (this implies there are no outgoing transitions).

Thus, $L(M) = \{w : \text{ There exists a tree as above with root } q_0 w \}$

What about accepting states?

- We don't need/use them! We keep F for compatibility with the standard definition.
- However, if we had acceptance stati, then
  - An existential ID with no successors would never be accepting.
  - A universal ID with no successors would be accepting.
  - Note that now IDs are accepting/rejecting, not states.
  - Each ID in a tree as above would be accepting.

How about loops?

- We require our tree to be finite, so we can't loop forever.
- The definition above does not require to "stick with decisions", i.e., any ID (both existential and universal) could occur several times. This is not a problem (since the tree is still finite), but we can cut out these "finite loops" by making decisions for the existential IDs that lead to the leafs earlier.

Informal Example: Generalised Geography

Solving via ATM.

- On tape: Graph and designated node.
- Two states, q_0 (initial and <u>existential</u>) and q_1 (<u>universal</u>)
  (Omitting intermediate states that are needed to change designated node.)
- From one state to another: replace designated node by successor in graph.

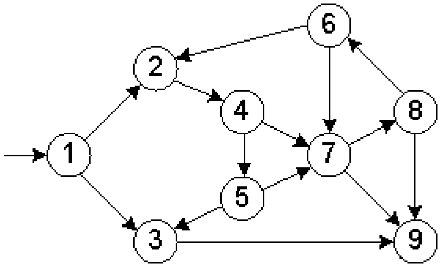
Explanation.

- IDs with q_0 are the states where player 1 moves, and IDs with q_1 are states where player 2 moves.
- If an ID with state q_0 doesn't have outgoing transitions: Player 1 loses.
- If an ID with state q_1 doesn't have outgoing transitions: Player 2 loses
  (as this is part of a winning strategy for player 1).

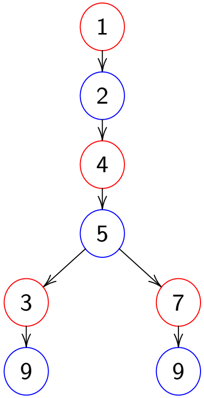
We'll re-visit this algorithm more formally in a few slides!

Informal Example: Generalised Geography, cont'd

Geography Graph.



Winning Strategy.



- existential states are red, universal states are blue
- In general, existential states and universal states don't have to alternate! Here we have this since we use the ATM for solving a turn-taking 2-player game.

First (ATM) Algorithm for Geography

```
Algorithm Geography (Graph G, start node n):
  let cur = n;
  forever do {
    existentially guess (a successor node e of cur);
    // if this is not possible, we don’t accept

    universally guess (a successor node u of e);
    // if there are no successors, we accept

    let cur := u;
  }
```

Comments.

- This shows (modulo a translation to TM) that Geography is solvable using an ATM.
- However the number of steps that this ATM takes is possibly infinite if there are loops in the graph. But looping means we neither accept nor reject, but we know that the problem should be (is) decidable! (We’ll revisit this Algorithm.)

Restrictions of ATMs

Definition. An ATM is polytime bounded if there exists a polynomial p such that every sequence of IDs from an initial ID (q_0, w) is at most $p(|w|)$ steps long.

The class **AP** of alternating polytime languages is the class of languages accepted by an ATM that is polytime bounded.

Observation.

- **NP** $\subseteq$ **AP** (because we only need existential states)
- **co-NP** $\subseteq$ **AP** (because we only need universal states)
- Both will also follow directly because – spoiler – we are going to show **AP** $=$ **PSPACE**, and both statements are known with regard to **PSPACE**.

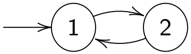
Reductions.

- Recall Theorem 10.1.7., week 7: If B is **NP**-hard and $B \leq_{\mathbf{P}} C$, then C is **NP**-hard.
- Recall Theorem 10.1.5., week 7: If $A \leq_{\mathbf{P}} B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- Now: If $L \leq_{\mathbf{P}} L'$ and $L' \in \mathbf{AP}$, then $L \in \mathbf{AP}$.

Example Revisited: Geography

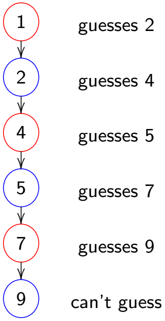
Earlier Algorithm.

```
Algorithm Geography (Graph G, start node n):
  let cur = n;
  forever do {
    existentially guess (a successor node e of cur);
    // if this is not possible, we don't accept

    universally guess (a successor node u of e);
    // if there are none, we accept

    let cur := u;
  }
```

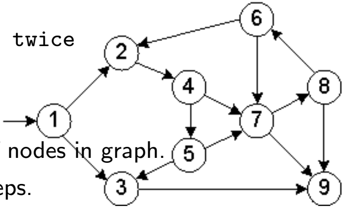
- not necessarily terminating, e.g.  (assume "fitting" transitions)
- let alone in polynomially many steps!

Geography, Terminating

Idea. Universal nodes don't need to repeat. (I.e., the Ex.-player doesn't play into loops.) (Existential nodes also don't have to, but it doesn't hurt either!)



```
Algorithm Geography2 (Graph G, start node cur):
  let seen := { cur };
  forever do {  // Player 1:
    existentially guess (cur := unseen successor of cur)
    // if this fails, we terminate and don't accept

    // Player 2:
    universally guess (cur := successor of cur);
    // if this fails, we terminate and accept

    seen := seen ‘union’ { cur } // never visit twice
  }
```



Geography in AP.

- Branches of tree at most twice as long as number of nodes in graph.
- Every computation path takes polynomially many steps.

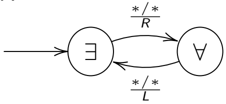
AP vs **co-AP**

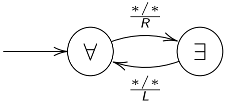
Observation. Given polytime bounded ATM M, construct ATM M' by swapping existential and universal states

- then M' accepts w if and only if M rejects w
- requires that all runs are terminating

Corollary. **co-AP** = **AP** (Again, this also follows from **AP** = **PSPACE**)

Example. What are the strings accepted by the TM and its dual version below, where * indicates any letter?





Their languages, in both cases, is $\emptyset$, because we never can't generate the desired proof tree with universal leaf nodes.

Exercise. Construct a simple ATM (with 2 states) that terminates on all runs and check above's claim. E.g., one that only runs to the right until the first blank is found and changes the input word.

QBF Revisited

Idea. $\exists \rightsquigarrow$ existential guess, $\forall \rightsquigarrow$ universal guess

```
Algorithm evalqbf (formula A):
  case A of {
    A_1 OR A_2: if (evalqbf A_1) = 1 then 1 else evalqnf(A_2)
    A_1 AND A_2: if (evalqbf A_1) = 0 then 0 else evalqbf(A_2)
    NOT A_1 : return 1 - (evalqnf A_1)
    exists x A : existentially guess v in {0, 1};
                 evalqbf A [ x := v]
    forall x A : universally guess v in {0, 1};
                 evalqbf A [ x := v]
  }
```

where `A [x := v]` replaces all free occurrences of `x` in `A` with `v`.

Theorem.

- QBF is in **AP** (by algorithm above)
- **PSPACE** $\subseteq$ **AP** (as QBF is **PSPACE**-hard)

From **AP** to **PSPACE**

Theorem. **AP** $\subseteq$ **PSPACE**.

Proof (Idea). Depth-first search simulates ATM M on standard TM.

```
Algorithm ATMaccept (ATM-ID I):
  if (I is existential) {
    let accept := false;
    foreach J with I |- J { accept := accept OR ATMaccept(J); }
    return (accept);
  } else if (I is universal) {
    let accept := true;
    foreach J with I |- J { accept := accept AND ATMaccept(J); }
    return (accept);
  }
```

For polynomial bound p and input of length n:

- recursion depth is polynomial as M is **AP**
- argument in recursive calls is of size $O(p(n))$

So space in $O(p^2(n))$.

Why **PSPACE** is "harder" than **NP**

- True **NP**-complete instances can (at least) be easily verified:
  Provide witness (e.g., accepting-ID-path of NTM).
  Has polynomial length and can be verified in polynomial time.
- Example: Is a SAT formula satisfiable?
  Verifier can check correctness of variable assignment in polytime.
- True **PSPACE**-complete problems can (probably) not be easily verified:
  Prover even of unlimited power cannot convince poly-time verifier that some language is in some class (if **PSPACE** $\neq$ **NP**). (Because **PSPACE** $\neq$ **NP** directly, due to our central corollary, excludes a polytime witness.)

What we know and don't know (Recap)

Inclusions

- $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} = \mathbf{NPSPACE} \subseteq \mathbf{EXPTIME} \subseteq \mathbf{NEXPTIME}$
- but $\mathbf{P} \neq \mathbf{EXPTIME}$, but we don't know which inclusion(s) is (are) strict

Co-Classes.

- $\mathbf{co\text{-}PSPACE} = \mathbf{PSPACE} = \mathbf{NPSPACE} = \mathbf{co\text{-}NPSPACE}$
- $\mathbf{NP} \subseteq \mathbf{PSPACE}$ and $\mathbf{co\text{-}NP} \subseteq \mathbf{PSPACE}$
- but we don't know whether $\mathbf{NP} = \mathbf{co\text{-}NP}$ (however this would follow if $\mathbf{P} = \mathbf{NP}$)

Equalities (from Today)

- $\mathbf{PSPACE} = \mathbf{AP}$