COMP3630 / COMP6363

week 11: **Bonus Lecture**

slides created by: Pascal Bercher

convenor & lecturer: Pascal Bercher

The Australian National University

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Content of this Chapter

- Hardness with “powerful” reductions.
- Recap on stati of Turing Machines

On **P** membership vs. **P**-completeness

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Idea.

- We can use the polytime granted by the reduction to solve the other problem!
- We'll see in the proof (next slide) why trivial problems don't work.

P-hardness of non-trivial **P** problems, Proof

- Let $L \in \mathbf{P}$ and $L \neq \emptyset$ and $L \neq \Sigma^*$.
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- To show **P**-hardness, we prove that "it can be used" (under Karp-reductions!) to solve any other problem in **P**. In other words: for any $L' \in \mathbf{P}$ we get $L' \leq_{\mathbf{P}} L$
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- w_{yes}/w_{no} are fixed in advance, "hard-coded" into the function. So the runtime for writing the word $f(w)$ is <u>constant</u> and does not scale with the length of w! (f however still needs non-constant poly-time to check for $w \in L'$.)

P-completeness, final notes

So, why did we never talk about **P**-completeness?

- Because we only covered Karp-reductions, for which he have that **P** membership implies **P**-hardness.
- Note that **P**-hardness/completeness is not defined using Karp-reductions! (Now you see why!)
- They are based on log-time reductions to differentiate between problems in **P** and those (believed to be?) below. This goes beyond this course, just remember that **P**-hardness is defined differently than used in this course!

Why polytime-reductions for **PSPACE**-hardness?

Recall last week (week 10), chapter 11c. We asked why we define **PSPACE**-hardness via poly-time reductions, rather than poly-space reductions. Now you know why!

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Proof.
Identical to the one before. Just replace **P** by **PSPACE** and poly-time by poly-space.

Disclaimer

- The following slides are exactly the same as those in week 5, pages 9 to 11.
- Thus they will be excluded here from the upload to prevent redundancy.