COMP3630 / COMP6363

week 1: **Finite Automata**

This Lecture Covers Chapter 2 of HMU: Finite Automata

slides created by: Dirk Pattinson, based on material by
Peter Hoefner and Rob van Glabbeck; with improvements by Pascal Bercher

convenor & lecturer: Pascal Bercher

The Australian National University

Semester 1, 2023

COMP3630/6363: Theory of Computation

Textbook. *Introduction to Automata Theory, Languages and Computation* by John E. Hopcroft, Rajeev Motwani, and Jeffrey D. Ullman [HMU].

Prerequisites. Chapter 1 of HMU (sets, functions, relations, induction) (if you prefer lectures over reading, I uploaded one on YouTube)

Assessment.
- 3 assignments — each @ 12%
- 1 online quiz (1 hour) in week 4 — @ 4%
- 1 final exam — @ 60%

Labs. Participation is voluntary, but highly recommended.
- Thursday, 11 am to 1 pm and Friday, 12 pm to 2 pm
- each tutorial covers the content of the same week
- tutor of weeks 1 to 6: Adam Rowland
- tutor of weeks 7 to 12: Timothy Horscroft

Content. Languages / Automata / Computability / Complexity
This course is basically an advanced Math course.

Convenor. Pascal (Bercher), pascal.bercher@anu.edu.au

Lecturer. same!

Slides. Dirk Pattinson

CECC Class Representatives

Class Student Representation is an important component of the teaching and learning quality assurance and quality improvement processes within the ANU College of Engineering and Computer Science (CECC).

The role of Student Representatives is to provide ongoing constructive feedback on behalf of the student cohort to Course Conveners and to Associate Directors (Education) for continuous improvements to the course.

Roles and responsibilities:

- Act as the official liaison between your peers and convener.
- Be available and proactive in gathering feedback from your classmates.
- Attend regular meetings, and provide reports on course feedback to your course convener
- Close the feedback loop by reporting back to the class the outcomes of your meetings.

Why become a class representative?

- **Ensure students have a voice** to their course convener, lecturer, tutors, and College.
- **Develop skills sought by employers**, including interpersonal, dispute resolution, leadership and communication skills.
- **Become empowered**. Play an active role in determining the direction of your education.
- **Become more aware of issues influencing your University** and current issues in higher education.
- **Course design and delivery.** Help shape the delivery of your current courses, as well as future improvements for following years.

Note: Class representatives will need to be comfortable with their contact details being made available via Wattle to all students in the class.

For more information regarding roles and responsibilities, contact:

ANUSA CECC representatives: sa.cecs@anu.edu.au

Want to be a class representative? Nominate today!

Please nominate yourself to your course convener by end of Week 2, Sem 1, 2023.

You are free to nominate yourself whether you are currently on-campus or studying remotely.

Class Representatives

Interested?

- Up to two class reps per course code (COMP1600 / COMP6260)
- Self-nominate yourself by dropping Pascal an email
- Selections must be final by the end of week 2

Content of this Chapter

- Deterministic Finite Automata
- Nondeterministic Finite Automata
- NFA with ϵ-transitions
- An Equivalence among the above three.

(This was all covered in COMP1600)

Additional Reading: Chapter 2 of HMU.

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- A **(formal) language** is a subset of Σ^*.

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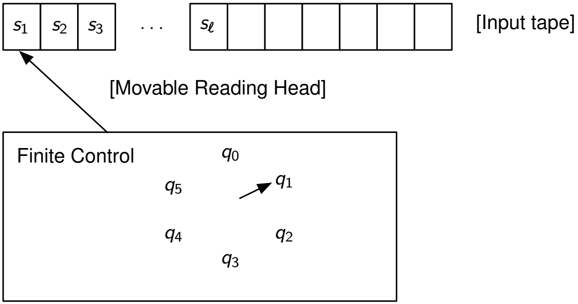
Informally:



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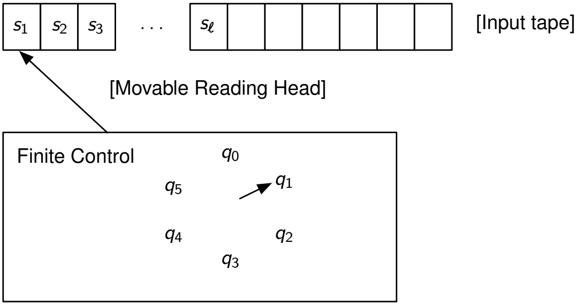

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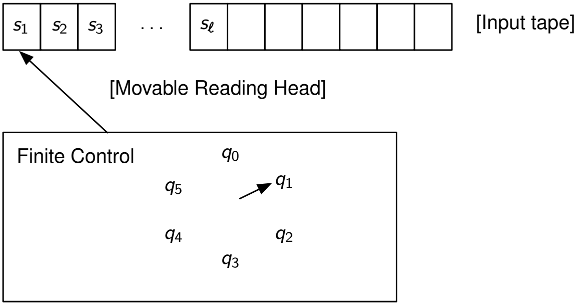

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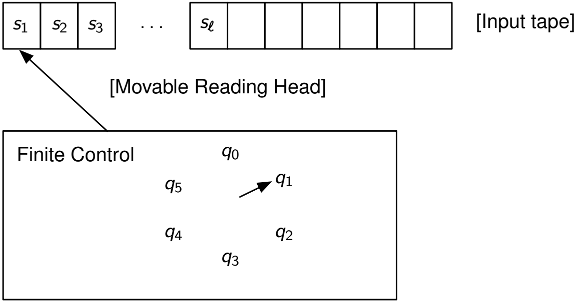

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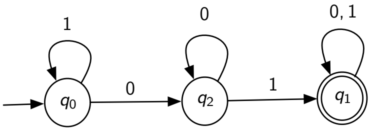
Transition Table:

	0	1
$\rightarrow q_0$	q_2	q_0
$* q_1$	q_1	q_1
q_2	q_2	q_1

$F = \{q_1\}$

$$\delta(q_0, 0) = q_2$$
$$\delta(q_0, 1) = q_0$$

Transition Diagram:



Language accepted by a DFA

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In other words:

- $\epsilon \in L(A) \Leftrightarrow q_0 \in F$
- For $k > 0$,

$$w = s_1 s_2 \cdots s_k \in L(A) \Leftrightarrow q_0 \xrightarrow{s_1} P_1 \xrightarrow{s_2} P_2 \xrightarrow{s_3} \cdots \xrightarrow{s_k} P_k \in F$$

An Example

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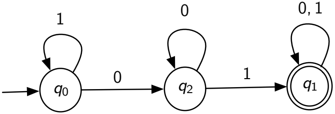

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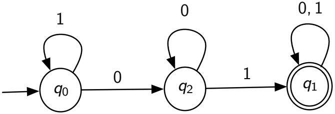

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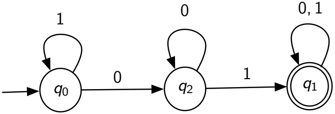

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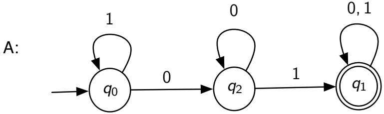

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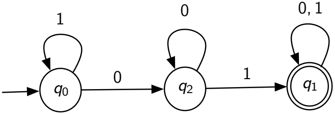

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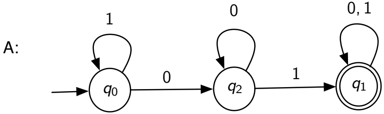

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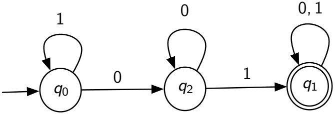

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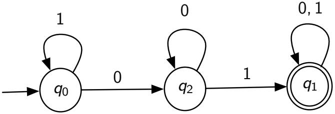

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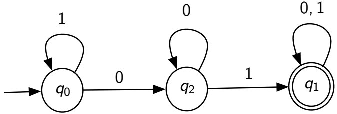

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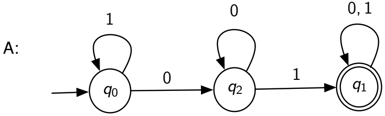

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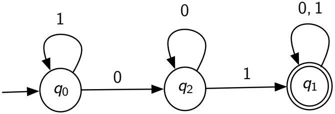

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- Remark 2: Multiple strings can have the same path of states. For example, 0010 and 0011 have the same sequence of states.

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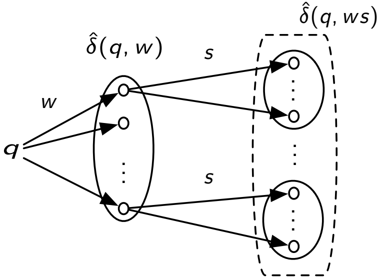

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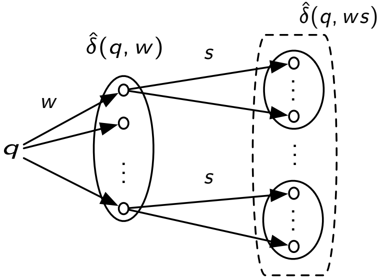

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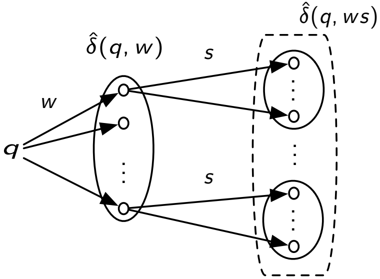

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In other words:

- $\epsilon \in L(A) \Leftrightarrow q_0 \in F$
- For $k > 0$,

  $$w = s_1 s_2 \cdots s_k \in L(A) \Leftrightarrow \exists \text{ a path } q_0 \xrightarrow{s_1} P_1 \xrightarrow{s_2} P_2 \xrightarrow{s_3} \cdots \xrightarrow{s_k} P_k \in F$$

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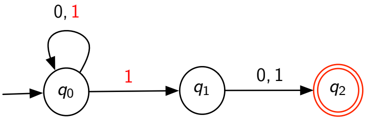

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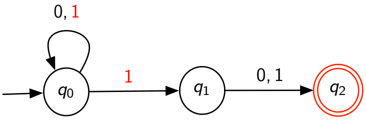

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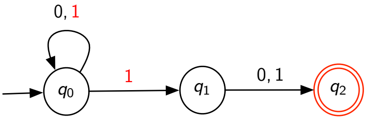

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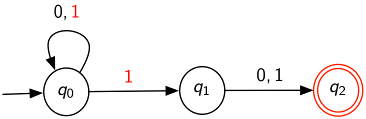

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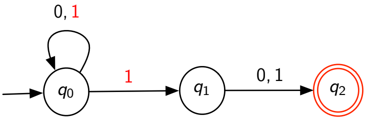

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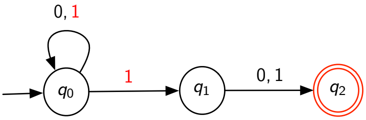

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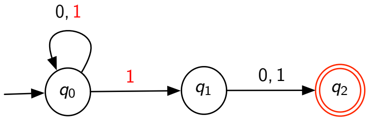

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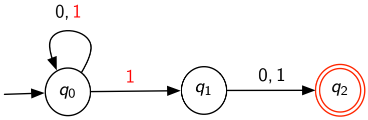

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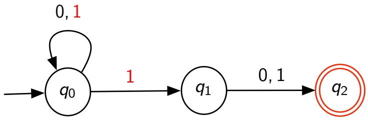

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Theorem 2.4.1

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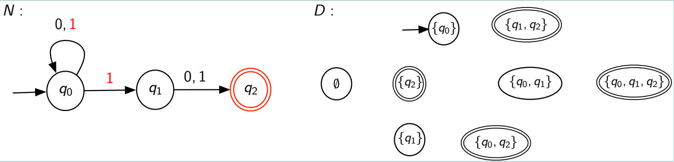

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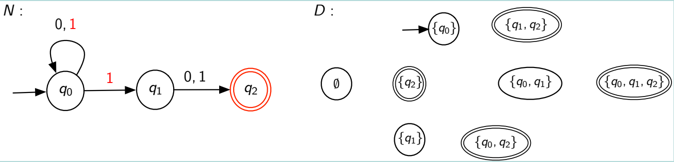

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- Hence, $\epsilon \in L(N) \;\Leftrightarrow\; q_0 \in F_N \;\Leftrightarrow\; \{q_0\} \in F_D \;\Leftrightarrow\; \epsilon \in L(D)$

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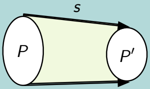
N: 

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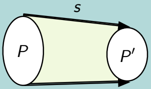

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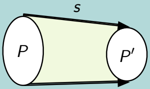

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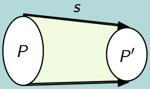

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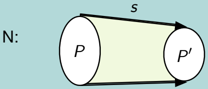

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- Thus, $\hat{\delta}_N(q_0, \cdot) = \hat{\delta}_D(\{q_0\}, \cdot)$, and hence the languages have to be identical.

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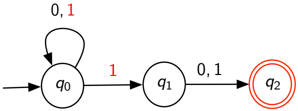

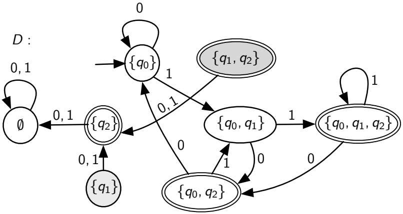

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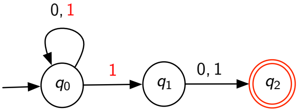

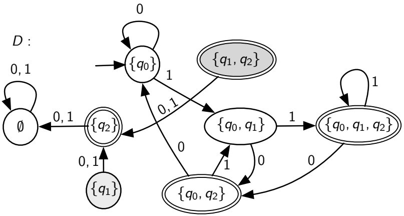

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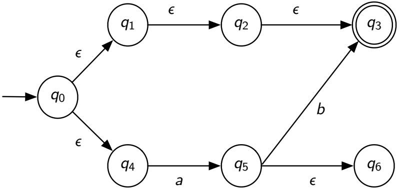

	ϵ	a	b
$\rightarrow q_0$	$\{q_1, q_4\}$	$\emptyset$	$\emptyset$
q_1	$\{q_2\}$	$\emptyset$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\emptyset$
$* q_3$	$\emptyset$	$\emptyset$	$\emptyset$
q_4	$\emptyset$	$\{q_5\}$	$\emptyset$
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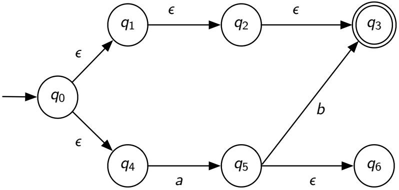

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q_6	$\emptyset$	$\emptyset$	$\emptyset$

- Without reading any input symbols, the state of the ϵ-NFA can transition:

From q_0 to q_1, q_4, q_2, or q_3. From q_1 to q_2, or q_3.

From q_2 to q_3. From q_5 to q_6.

Language Accepted by an ϵ-NFA

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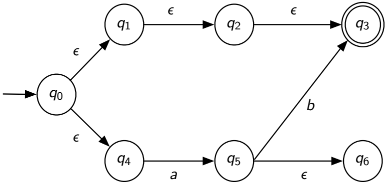

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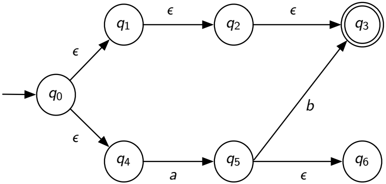

$$
\begin{aligned}
\texttt{ECLOSE}(q_0) &= \{q_0, q_1, q_4, q_2, q_3\} \\
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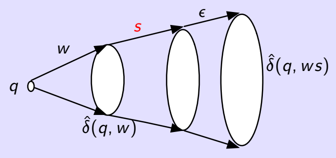

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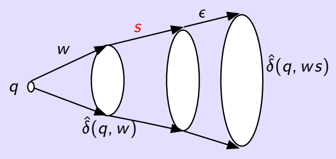

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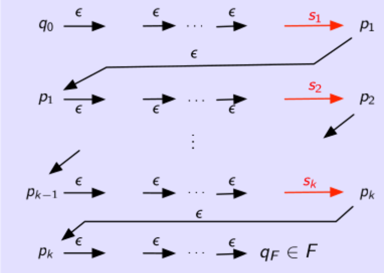
  - For $k > 0$, $w = s_1 s_2 \dots s_k \in L(N) \Leftrightarrow \exists$ a path such as the following:



Do ϵ-NFAs Accept More Languages?

Theorem 2.5.1

Every Language L that is accepted by an ϵ-NFA is also accepted by some DFA.

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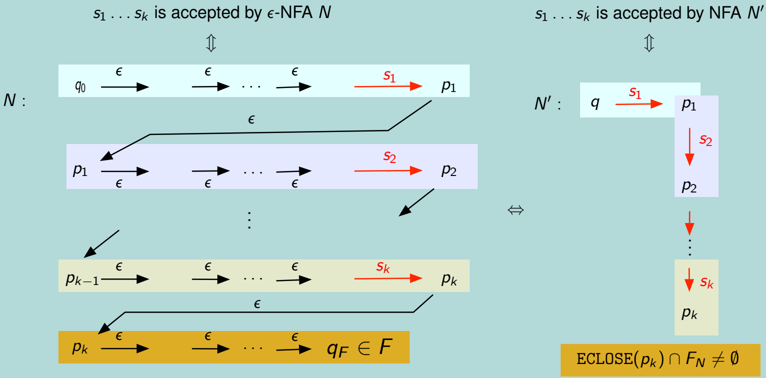
Proof of Theorem 2.5.1 (Abstract idea)



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$$N: \quad q \xrightarrow{\epsilon} \quad \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} p \xrightarrow{s} p'$$

N : q can transition to p' after a few ϵ-transitions, and a single read of $s \in \Sigma$.

$$\Updownarrow$$

$$N': \quad q \xrightarrow{s} p'$$

N': q can transition to p' after reading s.

To Summarize...

Languages accepted by DFAs	=	Languages accepted by NFAs	=	Languages accepted by ϵ-NFAs

- Allowing non-determinism and/or ϵ-transitions does not improve the language acceptance power of (finite) automata.