COMP3630 / COMP6363

week 1: **Regular Expressions and Languages**

This Lecture Covers Chapter 3 of HMU: Regular Expressions and Languages

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The Australian National University

Semester 1, 2023

Content of this Chapter

- Introduction to regular expressions and regular languages
- Equivalence of classes of regular languages and languages accepted
- Algebraic laws of (abstract) regular expressions

Additional Reading: Chapter 3 of HMU.

Regular Expressions: Overview

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- Each regular expression r denotes a language $L(r) \subseteq \Sigma^*$

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- Only those generated by the above induction are regular.
- **Remark**: Some authors/texts use | instead of $+$. HMU uses $+$.
- Precedence Rules:

$$(\cdot) > * > \cdot > +$$

where $>$ is 'binds more strongly than', and both $+$ and $\cdot$ associate to the left.

Regular Expressions: Examples

- $r = 0 + 11^*10$ is a regular expression
  - with brackets that indicate precedence: $r = 0 + (1(1^*)10)$
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  - with brackets that indicate precedence: $r = 0 + (1(1^*)10)$
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- Computing the language:

$$
\begin{aligned}
L(r) &= L(0) \cup L(11^*10) \\
&= \{0\} \cup L(1) \cdot L(1^*) \cdot L(1) \cdot L(0) \\
&= \{0\} \cup \{1\} \cdot \{1\}^* \cdot \{1\} \cdot \{0\} \\
&= \{0\} \cup \{1\} \cdot \{1^n \mid n \geq 0\} \cdot \{1\} \cdot \{0\} \\
&= \{1^i 0 \mid i \neq 1\}
\end{aligned}
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- Q: What's a regular expression that describes alternating sequences of 0s and 1s?

Regular Languages: Some Basic Properties

Theorem 3.2.1

Let $w \in \Sigma^*$. Then $\{w\}$ is regular.

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- $\{w\}$ being regular means there is a regular expression r with $L(r) = \{w\}$.
  Proof by induction on the length of w. For $w = \epsilon$, $\{w\} = \{\epsilon\} = L(\epsilon)$. For w of the form $w's$, we have (by induction) r s.t. $\{w'\} = L(r)$ so that $\{w\} = \{w's\} = L(rs)$.

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Let L_1 and L_2 be regular languages. Then, L_1^*, $L_1 \cup L_2$ and $L_1 L_2$ are also regular.

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- **Corollary 1:** The class of regular languages is closed under finite union and concatenation, i.e., if $L_1, \ldots, L_k$ are regular languages for any $k \in \mathbb{N}$, then $L_1 \cup \cdots \cup L_k$ and $L_1 \cdots L_k$ are also regular languages.

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- **Corollary 2:** Any finite language is regular.

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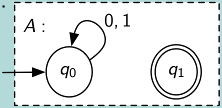
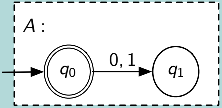


ϵ



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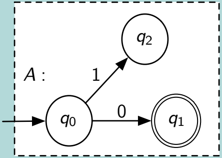

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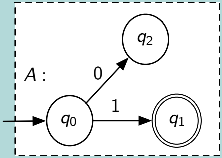

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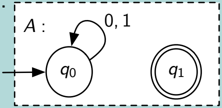
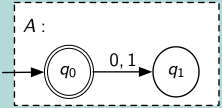


ϵ



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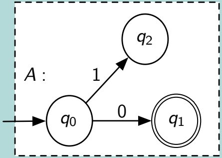

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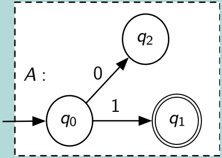


Note that these automata could be made smaller:

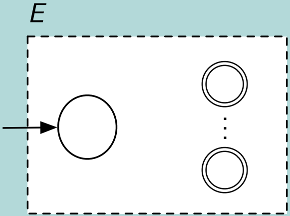
$\emptyset/\epsilon$ only keep initial state and no transitions since runs with non-existent transitions fail.

$0/1$ q_2 can be removed since runs with non-existent transitions fail.

DFAs and Regular Languages

Proof of Theorem 3.2.3 (Cont'd)

- Induction E^*:



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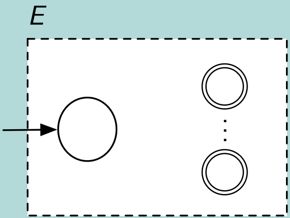

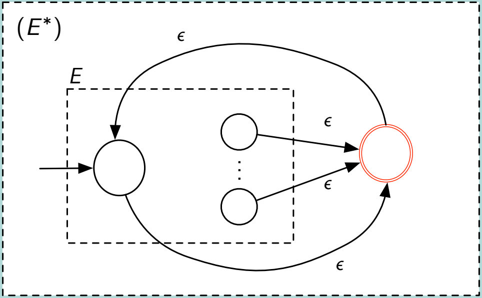

DFAs and Regular Languages

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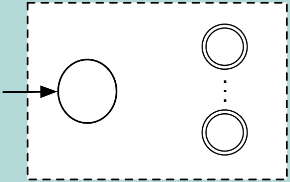
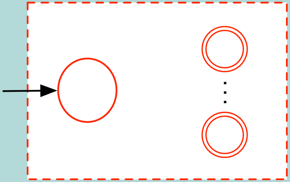
E



F



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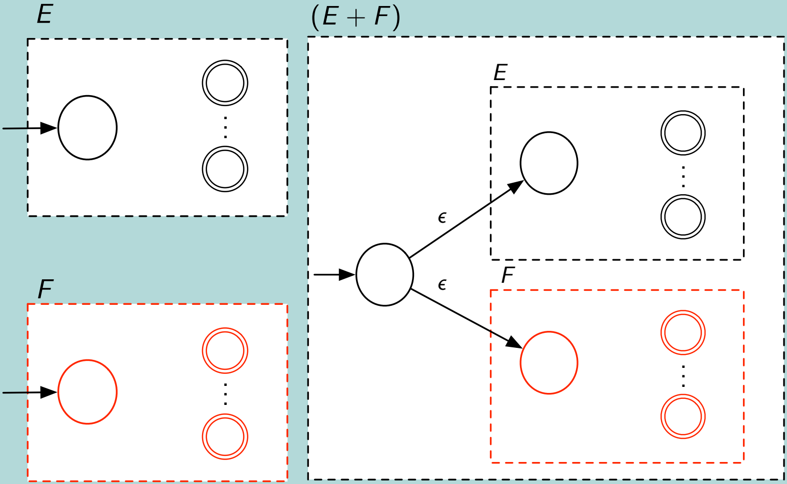

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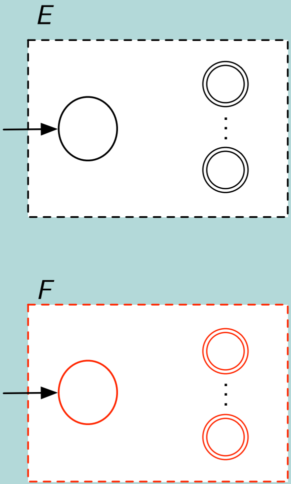

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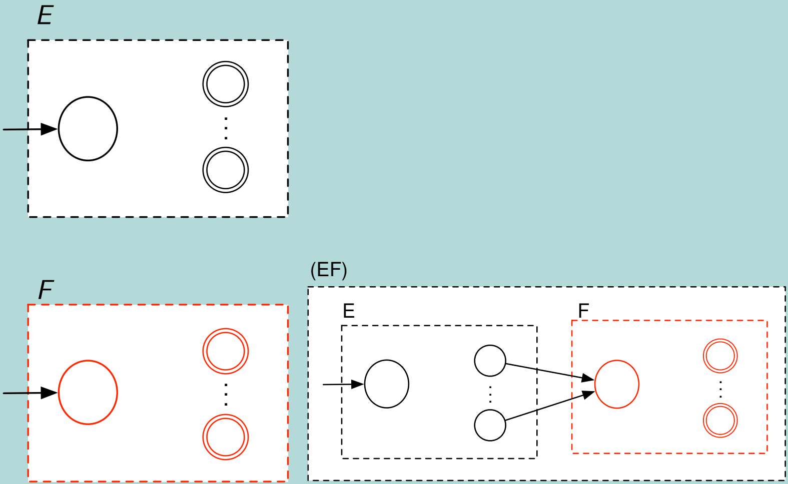
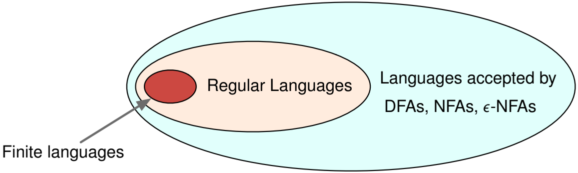


So Far...


Regular Languages
Languages accepted by
DFAs, NFAs, ϵ-NFAs
Finite languages

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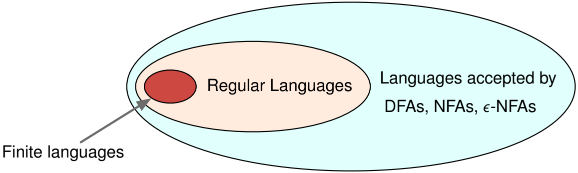

- Is the inclusion strict?
- Are there languages accepted by DFAs that are not regular?

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Proof of Theorem 3.2.4

- Let DFA $A = (Q, \Sigma, \delta, q_0, F)$ be given.
- Let us rename the states so that $Q = \{q_0, q_1, q_2, \ldots, q_{n-1}\}$.

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DFAs and Regular Languages

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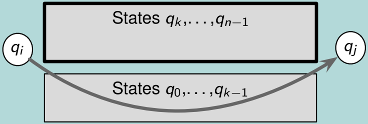

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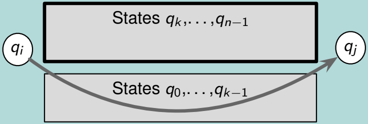

- Idea: prove that (a) each $R(i, j, k)$ is regular, and (b) $L(A)$ is a union of $R(i, j, k)$'s.

DFAs and Regular Languages

Proof of Theorem 3.2.4 (Cont'd)

- Note that $L(A) = \bigcup_{j:q_j \in F} R(0, j, n)$. (i.e., paths that start in q_0 and end in an accepting state with intermediate nodes $q_0, q_1, \ldots, q_{n-1}$ (all nodes))

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DFAs and Regular Languages

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DFAs and Regular Languages

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DFAs and Regular Languages

Proof of Theorem 3.2.4 (Cont'd)

- The strings in $R(i, j, k)$ correspond to paths whose intermediate nodes belong to $\{q_0, \ldots, q_{k-1}\}$.

DFAs and Regular Languages

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DFAs and Regular Languages

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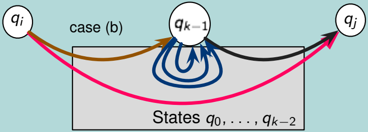

DFAs and Regular Languages

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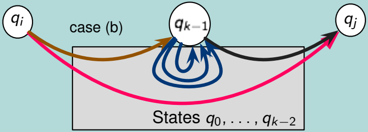

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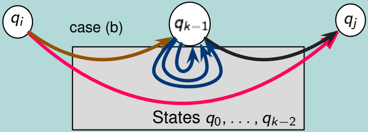

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DFAs and Regular Languages

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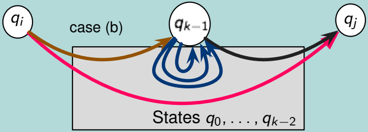

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DFAs and Regular Languages

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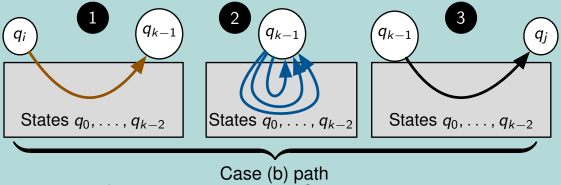

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DFAs and Regular Languages

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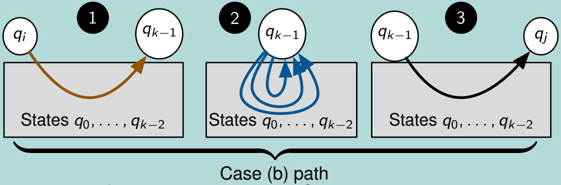

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DFAs and Regular Languages

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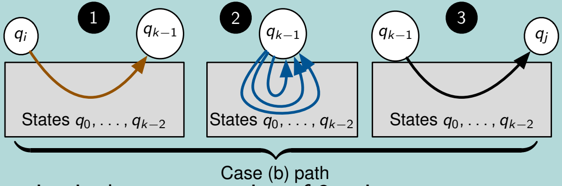

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DFAs and Regular Languages

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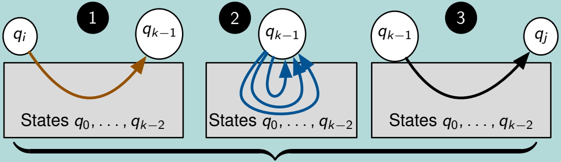

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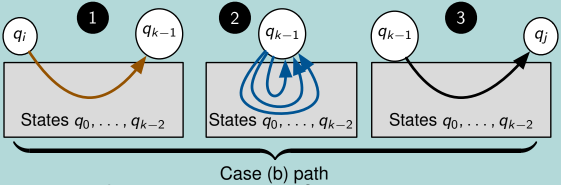

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  Thus,

$$R(i,j,k) = R(i,j,k-1) \cup [R(i,k-1,k-1)R(k-1,k-1,k-1)^* R(k-1,j,k-1)]$$

- From Thm 3.2.2, it follows that $R(i,j,k)$ is regular for any i, j, k. Thus, $L(A)$ is regular.

Equivalence of Languages

- The following are indeed equivalent:
  - The class of regular languages
  - The class of languages accepted by DFAs
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- We can introduce a notion of equality of (abstract) regular expression:

Abstract regular expressions $E_1 = E_2 \Leftrightarrow$ For any assignment of languages to the variables contained in E_1, E_2, their evaluations equal (i.e., $L(E_1) = L(E_2)$)

Algebraic Laws of Abstract Regular Expressions

- Commutativity: $\mathtt{L} + \mathtt{M} = \mathtt{M} + \mathtt{L}$ (Union is commutative)
  $\mathtt{LM} \neq \mathtt{ML}$ (Concatenation is not commutative)
- Associativity: $(\mathtt{L} + \mathtt{M}) + \mathtt{N} = \mathtt{L} + (\mathtt{M} + \mathtt{N})$ (Union is associative)
  $(\mathtt{LM})\mathtt{N} = \mathtt{L}(\mathtt{MN})$ (Concatenation is associative)
- Identity: $\emptyset + \mathtt{L} = \mathtt{L} + \emptyset = \mathtt{L}$ ($\emptyset$ is the identity element for $+$)
  $\epsilon\mathtt{L} = \mathtt{L}\epsilon = \mathtt{L}$ (ϵ is the identity element for concatenation)
- Annihilator: $\emptyset\mathtt{L} = \mathtt{L}\emptyset = \emptyset$
- Idempotent: $\mathtt{L} + \mathtt{L} = \mathtt{L}$
- Distributive: $\mathtt{L}(\mathtt{M} + \mathtt{N}) = \mathtt{LM} + \mathtt{LN}$
  $(\mathtt{M} + \mathtt{N})\mathtt{L} = \mathtt{ML} + \mathtt{NL}$
- Kleene $*$: $(\mathtt{L}^*)^* = \mathtt{L}^*$; $\emptyset^* = \epsilon$; $\epsilon^* = \epsilon$.