COMP3630 / COMP6363

week 4: **Properties and Normal Forms of Context-free Languages**

This Lecture Covers Chapter 7 of HMU: Properties of Context-free Languages

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Content of this Chapter

- Chomsky Normal Form
- Pumping Lemma for Context-free Languages (CFLs)
- Closure Properties of CFLs
- Decision Properties of CFLs

Additional Reading: Chapter 7 of HMU.

Chomsky Normal Forms

- A **normal** or **canonical form** (be it in algebra, matrices, or languages) is a standardized way of presenting the object (in this case, languages).
- A normal form for CFGs provides a prescribed structure to the grammar without compromising on its power to define **all** context-free languages.
- **Motivation:** Such normal forms can be exploited by algorithms (don't have to deal with all possible cases) and by proofs (same reason: can exploit this structure).
- Every non-empty language L with $\epsilon \notin L$ has **Chomsky Normal Form** grammar $G = (V, T, \mathcal{P}, S)$ where every production rule is of the form:
  - $A \longrightarrow BC$ for $A, B, C \in V$, or
  - $A \longrightarrow a$ for $A \in V$ and $a \in T$.

  and every variable in V is useful, i.e. appears in the derivation of at least one terminal string: for all $X \in V$ there is α, β, w such that $S \overset{*}{\underset{G}{\Rightarrow}} \alpha X \beta \overset{*}{\underset{G}{\Rightarrow}} w$.
- CNF disallows:
  - ~~$A \longrightarrow \epsilon$~~ [$\epsilon$-productions].
  - ~~$A \longrightarrow B$~~ for $A, B \in V$. [Unit productions].
  - ~~$A \longrightarrow B_1 \cdots B_k$~~, $A \in V$, $B_i \in V \cup T$ for $k \geq 2$ [Complex productions].
- Note that CNF can also be provided if $\epsilon \in L$. We only need a few additional steps.

Towards CNF [Step 1: Remove ϵ-Productions]

The goal is to eliminate all ϵ-productions (see next slide for a definition).

Example: Grammar with ϵ-productions

Suppose $G = (\{A, B, C\}, \{0, 1\}, \mathcal{P}, A)$ with $\mathcal{P}$:

- $A \longrightarrow BC$
- $B \longrightarrow 0B \mid \epsilon$
- $C \longrightarrow C11 \mid \epsilon$

How could an equivalent grammar look like without ϵ-productions?

Example: Grammar without ϵ-productions (with same language as above)

Now, $G' = (\{A, B, C\}, \{0, 1\}, \mathcal{P}', A)$ with $\mathcal{P}'$:

- $A \longrightarrow BC \mid B \mid C \mid \not\epsilon$
- $B \longrightarrow 0B \mid 0 \mid \not\epsilon$
- $C \longrightarrow C11 \mid 11 \mid \not\epsilon$

Note that the ϵ is in the first language, but not in the second.

Towards CNF [Step 1: Remove ϵ-Productions]

- ϵ-production: $A \longrightarrow \epsilon$ for some $A \in V$.
- Let us call a variable $A \in V$ as **nullable** if $A \overset{*}{\underset{G}{\Rightarrow}} \epsilon$.
- We can identify nullable variables as follows:
  - Basis: $A \in V$ is nullable if $A \longrightarrow \epsilon$ is a production rule in $\mathcal{P}$.
  - Induction: $B \in V$ is nullable if $B \longrightarrow A_1 \cdots A_k$ is in $\mathcal{P}$, and **each** A_i is nullable.

Procedure to Eliminate ϵ-Productions

- Given $G = (V, T, \mathcal{P}, S)$ define $G_{\text{no-}\epsilon} = (V, T, \mathcal{P}_{\text{no-}\epsilon}, S)$ as follows:

1. Start with $\mathcal{P}_{\text{no-}\epsilon} = \mathcal{P}$. Find all nullable variables of G.
3. For each production rule in $\mathcal{P}$ do the following:
   - If the body contains $k > 0$ nullable variables, add $2^k - 1$ productions to $\mathcal{P}_{\text{no-}\epsilon}$ obtained by choosing all subsets of nullable variables and removing them
4. Delete any production in $\mathcal{P}_{\text{no-}\epsilon}$ of the form $Y \rightarrow \epsilon$ for any $Y \in V$.

Examples: Suppose that in a given grammar, B, D are nullable and C is not.

- If $A \longrightarrow BCD$ is a rule in $\mathcal{P}$, then $A \longrightarrow BCD|CD|BC|C$ are rules in $\mathcal{P}_{\text{no-}\epsilon}$.
- Similarly, if $A \longrightarrow BD$ is a rule in $\mathcal{P}$, then $A \longrightarrow BD|B|D$ are rules in $\mathcal{P}_{\text{no-}\epsilon}$.

Towards CNF [Step 1: Remove ϵ-Productions]

Examples

- The one from Slide 4. (Eliminates ϵ from language.)
- The two from Slide 5. (Languages stay equivalent.)

Theorem 7.1.1

The induction procedure described in Slide 5 identifies **all** *nullable variables.*

Theorem 7.1.2

$L(G_{no\text{-}\epsilon}) = L(G) \setminus \{\epsilon\}$.[a]

[a] Proof in the Additional Proofs Section at the end

Recall: We could extend the procudure to keep $\epsilon \in L(G)$.
Procedure: Add a new start symbol with two rules:

- One that goes into ϵ (only if $\epsilon \in L(G)$),
- one that goes into the original start symbol.

Towards CNF [Step 2: Remove Unit Productions]

Example: Grammar with Unit Productions

Suppose $G = (\{A, B, C, D\}, \{a, b\}, \mathcal{P}, A)$ with $\mathcal{P}$:

- $A \longrightarrow aC \mid B$
- $B \longrightarrow bD \mid A$
- $C \longrightarrow aC \mid \epsilon$
- $D \longrightarrow bD \mid \epsilon$

How could an equivalent grammar look like without unit productions?

Example: Grammar without Unit Productions

Suppose $G = (\{A, B, C, D\}, \{a, b\}, \mathcal{P}, A)$ with $\mathcal{P}$:

- $A \longrightarrow aC \mid bD$ ~~$\mid B$~~
- $B \longrightarrow bD \mid aC$ ~~$\mid A$~~
- $C \longrightarrow aC \mid \epsilon$
- $D \longrightarrow bD \mid \epsilon$

Note: Rules with B being the head can **never** be used.

Towards CNF [Step 2: Remove Unit Productions]

- Given a grammar G and variables $A, B \in V$, we say (A, B) form a **unit pair** if $A \overset{*}{\underset{G}{\Rightarrow}} B$ using unit productions alone.
- We can identify unit pairs as follows:
  - Basis: For each $A \in V$, (A, A) is a unit pair (since $A \overset{*}{\underset{G}{\Rightarrow}} A$).
  - Induction: If (A, B) is a unit pair, and $B \to C$ is a production in $\mathcal{P}$, then (A, C) is a unit pair.
- Note: Suppose $A \longrightarrow BC$ and $C \longrightarrow \epsilon$ are productions then $A \overset{*}{\underset{G}{\Rightarrow}} B$, but (A, B) is **not** a unit pair. (Though we are going to use this step after the first anyway.)

Procedure to Eliminate Unit Productions

- Given $G = (V, T, \mathcal{P}, S)$ define $G_{\text{no-unit}} = (V, T, \mathcal{P}_{\text{no-unit}}, S)$ as follows:
  1. Start with $\mathcal{P}_{\text{no-unit}} = \mathcal{P}$. Find all unit pairs of G.
  2. For every unit pair (A, B) and non-unit production rule $B \longrightarrow \alpha$, add rule $A \longrightarrow \alpha$ to $\mathcal{P}_{\text{no-unit}}$.
  3. Delete **all** unit production rules in $\mathcal{P}_{\text{no-unit}}$.

Towards CNF [Step 2: Remove Unit Productions]

Example

See Slide 7.

Theorem 7.1.3

The induction procedure on Slide 8 identifies **all** *unit pairs.*

Theorem 7.1.4

$L(G_{no\text{-}unit}) = L(G)$.[b]

[b] Outline of the proof is given in the Additional Proofs Section at the end

Towards CNF [Step 3: Remove Useless Variables]

- A symbol $X \in V \cup T$ is said to be
  - **generating** if $X \overset{*}{\underset{G}{\Rightarrow}} w$ for some $w \in T^*$;
  - **reachable** if $S \overset{*}{\underset{G}{\Rightarrow}} \alpha X \beta$ for some $\alpha, \beta \in (V \cup T)^*$; and
  - **useful** if $S \overset{*}{\underset{G}{\Rightarrow}} \alpha X \beta \overset{*}{\underset{G}{\Rightarrow}} w$ for some $w \in T^*$ and $\alpha, \beta \in (V \cup T)^*$.
    (Useful $\Rightarrow$ Reachable + Generating, but not necessarily vice versa!
    Suppose $X \overset{*}{\underset{G}{\Rightarrow}} a$, so X is generating. Assume $S \overset{*}{\underset{G}{\Rightarrow}} \alpha X \beta$, so X is reachable.
    Now assume each rule $A \longrightarrow \alpha$ with $X \in \alpha$ has another variabe $B \in \alpha$ with empty language. So we can't turn X into a terminal word, although X is generating!)
- Given a grammar G, we can identify generating variables as follows:
  - Basis: For each $a \in T$, $a \overset{*}{\underset{G}{\Rightarrow}} a$. So a is generating.
  - Induction: If $A \longrightarrow \alpha$, and every symbol of α is generating, so is A.
- Given a grammar G, we can identify reachable variables as follows:
  - Basis: $S \overset{*}{\underset{G}{\Rightarrow}} S$ so S is reachable.
  - Induction: If $A \longrightarrow \alpha$, and A is reachable, so is every symbol of α.

Towards CNF [Step 3: Remove Useless Variables]

Procedure to Eliminate Useless Variables

- Given $G = (V, T, \mathcal{P}, S)$ define $G_{\mathrm{G}} = (V_{\mathrm{G}}, T, \mathcal{P}_{\mathrm{G}}, S)$ as follows:
  - Find all generating symbols of G.
  - V_G is the set of all generating variables.
  - P_G is the set of production rules involving **only** generating symbols.
- Now, define $G_{\mathrm{GR}} = (V_{\mathrm{GR}}, T_{\mathrm{GR}}, \mathcal{P}_{\mathrm{GR}}, S)$ as follows:
  - Find all reachable symbols of G_{G}.
  - V_{GR} is the set of all reachable variables.
  - P_{GR} is the set of production rules involving **only** reachable symbols.

The Order of Eliminating Variables is Important!

- Consider $G = (\{A, B, S\}, \{0, 1\}, \mathcal{P}, S)$ with $\mathcal{P} :\ S \longrightarrow AB|0;\ A \longrightarrow 1A;\ B \longrightarrow 1.$
- A is not generating. Removing A and the rules $S \longrightarrow AB$ and $A \longrightarrow 1A$ results in B being unreachable. Removing B and $B \rightarrow 1$ yields $G_{\mathrm{GR}} = (\{S\}, \{0\}, S \longrightarrow 0, S)$.
- Reversing the order, we first see that all symbols are reachable; removing then the non-generating symbol A and production rules $S \longrightarrow AB$ and $A \longrightarrow 1A$ yields $G_{\mathrm{RG}} = (\{B, S\}, \{0\}, S \longrightarrow 0$ and $B \longrightarrow 0, S)$. But B is unreachable now!

Towards CNF [Step 3: Remove Useless Variables]

Theorem 7.1.5

The induction procedure on Slide 10 identifies **all** *generating variables.*

Theorem 7.1.6

The induction procedure on Slide 10 identifies **all** *reachable variables.*

Theorem 7.1.7

(1) $L(G) = L(G_{GR})$*; and*
(2) Every symbol in G_{GR} *is useful.*[c]

[c] Proof in the Additional Proofs Section at the end

Towards CNF [Step 4: Remove Complex Productions]

Procedure to Eliminate Complex Productions

- Given $G = (V, T, \mathcal{P}, S)$, define $\hat{G} = (\hat{V}, T, \hat{\mathcal{P}}, S)$ as follows:
  - Start with $\hat{G} = G$ and do the following operations.
  - For every terminal $a \in T$ that appears in the body of length 2 or more, introduce a new variable A and a new production rule $A \longrightarrow a$.
  - Replace the occurrence of all such terminals in the body of length 2 or more by the introduced variables.
  - Replace every rule $A \longrightarrow B_1 \cdots B_k$ for $k > 2$, by introducing $k-2$ variables $D_1, \ldots, D_{k-2}$, and by replacing the rule by the following $k-1$ rules:

$$\begin{array}{llll} A \longrightarrow B_1 D_1 & D_2 \longrightarrow B_3 D_3 & \cdots & D_{k-2} \longrightarrow B_{k-1} B_k \\ D_1 \longrightarrow B_2 D_2 & \cdots & D_{k-3} \longrightarrow B_{k-2} D_{k-2} & \end{array}$$

- Note: Each introduced variable appears in the head **exactly** once.

Theorem 7.1.8

$L(G) = L(\hat{G})$.[d]

[d] Outline of the proof is given in the Additional Proofs Section at the end

The Chomsky Normal Form

Theorem 7.1.9

For every context-free language L containing a non-empty string, there exists a grammar G in Chomsky Normal Form such that $L \setminus \{\epsilon\} = L(G)$.

Proof

- Since L is a CFL, it must correspond to some CFG G.
- Eliminate ϵ productions (Step 1) to derive a grammar G_1 from G such that $L(G_1) = L(G) \setminus \{\epsilon\}$.
- Eliminate unit productions (Step 2) to derive a grammar G_2 from G_1 such that $L(G_2) = L(G_1)$.
- Eliminate useless variables (Step 3) to derive a grammar G_3 from G_2 such that $L(G_3) = L(G_2)$.
- Eliminate complex productions (Step 4) to derive a grammar G_4 from G_3 such that $L(G_4) = L(G_3)$.
- G_4 contains no ϵ-productions, no unit productions, no useless variables, and all productions have one terminal or two non-terminals in the body; Hence G_4 is in CNF.

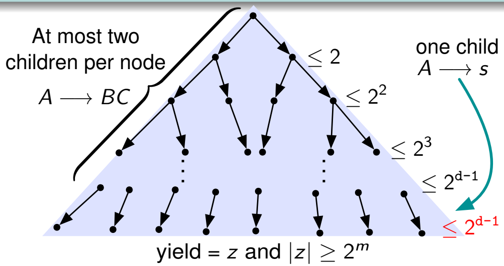
Pumping Lemma

Theorem 7.2.1

Let $L \neq \emptyset$ be a CFL. Then there exists $n > 0$ such that for any string $z \in L$ with $|z| \geq n$,
(1) $z = uvwxy$; (2) $vx \neq \epsilon$; (3) $|vwx| \leq n$; $uv^i wx^i y \in L$ for any $i \geq 0$.

Proof

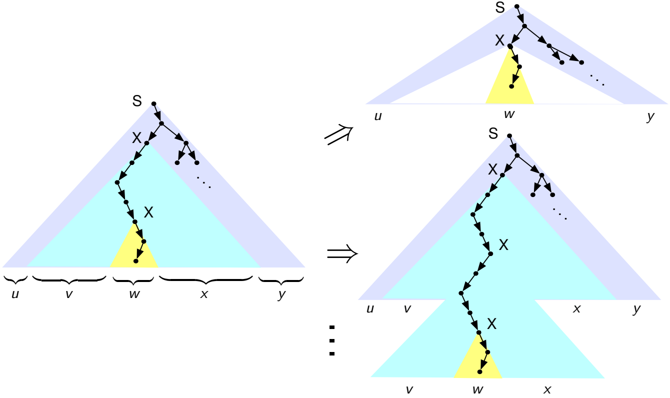
- Since the claim only pertains to non-empty strings, we can show the claim for $L \setminus \{\epsilon\}$.
- Let CNF grammar G generate $L \setminus \{\epsilon\}$. Choose $n = 2^m$ where $m = |V|$ in G.
- Pick any z with $|z| \geq n$.
- Depth $d \geq m + 1$.



Pumping Lemma

Proof

- Since depth $d \geq m+1$, there must be a path with with at least $m+1$ edges.
- In this path, at least two labels must match (since there are only $m = |V|$ many).
- Then, the Pumping Lemma claim follows from the following pictorial argument.

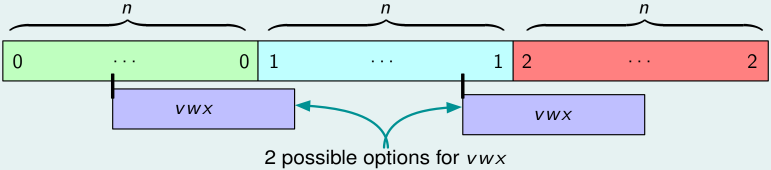


Uses of Pumping Lemma

- The Pumping Lemma (PL) can be used to argue that some langauges are **not** CFLs.

Proof that $L = \{0^n 1^n 2^n : n \geq 0\}$ is Not Context-Free

- Suppose it were.
- There exists an n such that for strings z longer than n pumping lemma applies.
- Applying the PL to $z = 0^n 1^n 2^n$, we see that $z = uvwxy$ such that $|vwx| \leq n$.



- vwx cannot contain both zeros and twos. Two cases arise:
  - Case (a): Suppose vwx contains no 2s. Then uwy contains fewer 1s or 0s than 2s. Such a string is not in L.
  - Case (b): Suppose vwx contains no 0s. Then uwy contains fewer 1s or 2s than 0s. Such a string is not in L.
- There are 3 cases with only one letter. (Also show that String is not in L.)

Substitution of Symbols with Languages

- Let L be a CFL on Σ_1, and let h be a **substitution**, i.e., for each $a \in \Sigma_1$, $h(a)$ is a language over some alphabet Σ_a.
- We can extend the substitution to words by concatenation, i.e., $h(s_1 \cdots s_k) = h(s_1)h(s_2) \cdots h(s_k)$. // here, we concatenate languages!
- One can then extend the substitution to languages by unioning, i.e.,

$$h(L) := \bigcup_{s_1 \cdots s_\ell \in L} h(s_1 \cdots s_\ell) = \bigcup_{s_1 \cdots s_\ell \in L} h(s_1) \cdots h(s_\ell)$$

  i.e., $h(L)$ is the language formed by substituting each symbol in a string in the language L by a corresponding language.

An Example

Let $h(a) = \{0\}$ and $h(b) = \{1, 11\}$, $L = \{a^n b^n : n \geq 0\}$, and $w = aabb \in L$. Then,

- $h(w) = h(a)h(a)h(b)h(b) = \{0\}\{0\}\{1, 11\}\{1, 11\} = \{00\}\{11, 111, 111, 1111\}$
  $= \{00\}\{11, 111, 1111\} = \{0011, 00111, 001111\} = \{0^2 1^m : 2 \leq m \leq 2 \cdot 2 = 4\}$
- $h(L) = \{0^n 1^m : n, m \geq 0, \text{ such that } n \leq m \leq 2n\}$

Substitution of Symbols with Languages

Theorem 7.3.1

If L is a CFL over Σ_1 and $h(a)$ is a CFL for every $a \in \Sigma_1$, then $h(L)$ is also a CFL.

Let G be a CFG generating L.

$$S \overset{*}{\Rightarrow} a_1 \cdots a_\ell \quad \text{(in } G\text{)} \qquad\qquad \text{For } i = 1, \ldots, \ell, \quad S_{a_i} \overset{*}{\Rightarrow} w_{a_i} \quad \text{(in } G_{a_i}\text{)}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$S \overset{*}{\Rightarrow} S_{a_1} \cdots S_{a_\ell} \quad \text{(in } \hat{G} \text{ as well as } G_{sub}\text{)} \qquad\qquad S_{a_i} \overset{*}{\Rightarrow} w_{a_i} \quad \text{(in } G_{sub}\text{)}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\overbrace{S \overset{*}{\Rightarrow} S_{a_1} \cdots S_{a_\ell}} \overset{*}{\Rightarrow} \overbrace{w_{a_1} S_{a_2} \cdots S_{a_\ell} \overset{*}{\Rightarrow} w_{a_1} w_{a_2} S_{a_3} \cdots S_{a_\ell} \overset{*}{\Rightarrow} \cdots \overset{*}{\Rightarrow} w_{a_1} \cdots w_{a_\ell}}$$

Substitution of Symbols with Languages

Theorem 7.3.1

If L is a CFL over Σ_1 and $h(a)$ is a CFL for every $a \in \Sigma_1$, then $h(L)$ is also a CFL.

Proof

- Let $G = (V, \Sigma_1, \mathcal{P}, S)$ be a grammar that generates L.
- For each $a \in \Sigma_1$, let $G_a = (V_a, \Sigma_a, \mathcal{P}_a, S_a)$ be a grammar that generates $h(a)$.
- WLOG, assume that $V \cap V_a = \emptyset$ for each $a \in \Sigma_1$.
- Now define $\hat{G} = (V, \{S_a : a \in \Sigma_1\}, \hat{\mathcal{P}}, S)$ by
  - Every rule of $\hat{\mathcal{P}}$ is a rule of $\mathcal{P}$ obtained by replacing each $a \in \Sigma_1$ by S_a.
  - For example, $X \to aXb$ in $\mathcal{P}$ will correspond to $X \to S_aXS_b$ in $\hat{\mathcal{P}}$ if $a, b \in \Sigma_1$.
- Let $G_{sub} = (V \cup (\cup_{a\in\Sigma_1} V_a), \cup_{a\in\Sigma_1} \Sigma_a, \hat{\mathcal{P}} \cup (\cup_{a\in\Sigma_1} \mathcal{P}_a), S)$. Claim: G_{sub} generates $h(L)$.
- Note that $w \in h(L)$ can be written as $w_{a_1} \cdots w_{a_\ell}$ for $w_{a_i} \in h(a_i)$ for each i, and for some $a_1 \cdots a_\ell \in L$.

Closure under substitution means...

Closure under:

- (Finite) Union: Let $L = \{1, 2, \ldots, k\}$ and $h(i) = L_i$ be a CFL for each $i = 1, \ldots, k$. By Theorem 7.3.1, $h(L) = L_1 \cup \cdots \cup L_k$ is a CFL.
- (Finite) Concatenation: Let $L = \{a_1 a_2 \cdots a_k\}$ and $h(a_i) = L_{a_i}$ be a CFL for each $i = 1, \ldots, k$. By Theorem 7.3.1, $h(L) = L_{a_1} \cdots L_{a_k}$ is a CFL.
- Kleene-$*$ closure: Let $L = \{a\}^*$ and $h(a) = L_a$ be a CFL. By Theorem 7.3.1, $h(L) = (L_a)^*$ is a CFL.
- Positive closure: Let $L = \{a\}^+ := \{a^n : n \geq 1\}$ and $h(a) = L_a$ be a CFL. By Theorem 7.3.1, $h(L) = L_a(L_a)^*$ is a CFL.
- Homomorphism: Let L be a CFL and g be a homomorphism (i.e., h maps symbols to strings of symbols over some alphabet). Define $h(a) = \{g(a)\}$, which is a regular/CF language.Then, $h(L) = g(L)$ and by Theorem 7.3.1, it is a CFL.

Some More Closure Properties – 1

Theorem 7.3.2

If L is CFL, then so is L^R.

Proof

- If $G = (V, T, \mathcal{P}, S)$ generates L, then $G^R = (V, T, \mathcal{P}^R, S)$ generates L^R where

$$A \to X_1 \cdots X_\ell \text{ in } \mathcal{P} \iff A \to (X_1 \cdots X_\ell)^R = X_\ell X_{\ell-1} \cdots X_1 \text{ in } \mathcal{P}^R$$

Theorem 7.3.3

If L is a CFL, R is a regular language, then $L \cap R$ is a CFL.

Proof of Theorem 7.3.3

- Product PDA approach: Run the PDA accepting L_1 and DFA accepting L_2 in parallel. Accept input string iff (if and only if) both machines accept.

Note: In an exam/paper/book, these would be proof ideas, not full proofs.

Some More Closure Properties – 2

Theorem 7.3.4

If L is a CFL and h is a homomorphism, $h^{-1}(L) = \{w : h(w) \in L\}$ is also a CFL.

A Coarse Outline of Proof of Theorem 7.3.4, Part 1

What we know:

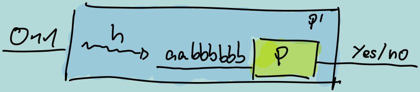
- $h : \Sigma_1 \to \Sigma_2$ (since homomorphisms can map to another alphabet)
- L is defined over Σ_2 and a CFL. Thus there is a PDA P with $L(P) = L$.

What we need:

- For $L' = h^{-1}(L)$ to be a CFL it suffices to show that there is a PDA P', such that: P' accepts w iff $h(w) \in L$, i.e., iff P accepts $h(w)$.

Example and Idea:

- Turn each w into $h(w)$, then use P.
- Let $\Sigma_1 = \{0, 1\}$, $\Sigma_2 = \{a, b\}$, $h(0) = aa$, $h(1) = bbb$, $w = 011$



Some More Closure Properties – 2 (cont'd)

A Coarse Outline of Proof of Theorem 7.3.4, Part 2

Problem:

- A PDA can't manipulate the input string! (Only Turing Machines can do that.)

Solution:

- We store the outcome of h in the state itself!
- Recall: $h(0) = aa$, $h(1) = bbb$.
- Let the states of PDA P be $q_0, \ldots, q_k$. Then, the PDA P' that accepts $h^{-1}(L)$ has $6k$ states, namely (q_i, aa), (q_i, a), (q_i, ϵ), (q_i, bbb), (q_i, bb), and (q_i, b).
- The transition between states of P' is defined as if the second component is the input tape (e.g., (q_i, aa) transitions to (q_i, a)). Once the second component is empty, we can move on reading another symbol from $\{0, 1\}$ and filling the second component again accordingly.

Some Non-Closure Properties

- CFLs are not closed under intersection.
  - Let $L_1 = \{0^n 1^n 2^m : n, m \geq 0\}$, $L_2 = \{0^n 1^m 2^n : n, m \geq 0\}$. Both are CFLs. However, $L_1 \cap L_2 = \{0^n 1^n 2^n : n \geq 0\}$ is not a CFL.
- CFLs are not closed under complementation.
  - Suppose CFLs are closed under complementation. Let L_1, L_2 be the aforementioned CFLs. Then $L_1 \cap L_2 = (L_1^c \cup L_2^c)^c$ **must** be a CFL (see slide 20), but it is not. Hence, CFLs cannot be closed under complementation.
  - Note: There exist CFLs L such that L^c is a CFL as well.
- CFLs are not closed under set difference.
  - Since CFLs are not closed under complementation, choose a CFL L such that L^c is not a CFL. But $L^c = \Sigma^* \setminus L$ and Σ^* is a CFL. Hence, CFLs are not closed under set difference.
  - Note: There exist CFLs L_1, L_2 such that $L_1 \setminus L_2$ is a CFL as well.

Language Emptiness

- Conversion of a grammar G to a corresponding PDA, PDA to a corresponding grammar G, and a grammar to CNF can each be achieved in polynomial time.

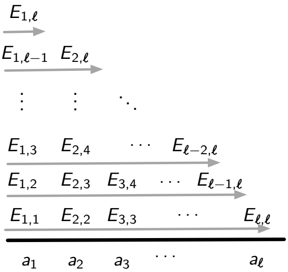
Emptiness of a CFL L

- Let a grammar $G = (V, T, \mathcal{P}, S)$ generating the language L be given. (If PDA is given, convert it to a grammar G).
- G is non-empty $\iff$ S is generating.

Language Membership – **The CYK Algorithm**

Membership of w in a CFL L

- Given CNF $G = (V, T, \mathcal{P}, S)$ and $w = a_1 \cdots a_\ell$ we identify $\sum_{i=1}^{\ell} i = \ell(\ell+1)/2 \in \mathcal{O}(\ell^2)$ sets $E_{i,j}$, with $1 \leq i \leq j \leq \ell$.
- $E_{i,j}$ corresponds to **all** variables that can derive $a_i \cdots a_j$.
- $E_{i,j}$'s are identified from bottom to top, left to right by the following induction.
  - Basis: For each $i = 1, \ldots, \ell$, $E_{i,i}$ contains **all** variables X such that $X \rightarrow a_i$.
  - Induction: For each $i = 1, \ldots, \ell$ and $j > i$, $E_{i,j}$ contains X if: (1) $X \longrightarrow YZ$ (2) $Y \in E_{i,i'}$ and $Z \in E_{i'+1,j}$ for some $i \leq i' \leq j$.
  - $S \in E_{1,\ell} \iff w \in L(G)$.



Some Undecidable Questions about CFGs and CFLs

You might not know yet what “Undecidable” means. Thus:

- You might want to get back to this in a few weeks!
- In a nutshell (and quite informally), it implies that there’s no algorithm that answers these questions correctly (with yes/no) and always terminates.

Undecidable questions:

- Is a given grammar unambiguous/ambiguous?
- Is a given CFL inherently ambiguous?
- Is the intersection of two CFLs empty?

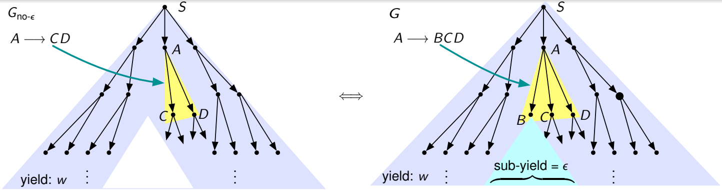
    (Fun fact: this is used to prove that HTN planning is undecidable.
    We might look into this in week 12!)
- Are two CFLs identical?
- Is a given CFL equal to Σ^*?

Additional Proofs

Proof of Theorem 7.1.2

- $\Leftarrow$ Construct a parse tree with yield $w \in L(G) \setminus \{\epsilon\}$. Identify a **maximal** subtree, rooted at say X, whose yield is ϵ. Delete X and its subtree. Repeat until no such subtrees remain. In this illustrative example below, suppose that there is only one subtree with ϵ yield; let B be its label and let $A \longrightarrow BCD$ be the production that introduced B. Now, delete B and its subtree. This new subtree is a parse tree for $G_{\text{no-}\epsilon}$ with yield w since $A \longrightarrow CD$ is a valid production rule in $\mathcal{P}_{\text{no-}\epsilon}$ [Why? B is nullable].
- $\Rightarrow$ Construct a parse tree with yield $w \in L(G_{\text{no-}\epsilon})$. Identify production rules (used in the tree) that are not in P. For each such rule, find an appropriate rule by appending nullable variables. To the parse tree, add the corresponding nullable variable(s) and a zero-yield subtrees to transform it to a parse tree for G.

  In the example, the portion of the parse tree in yellow corresponds to the rule $A \longrightarrow CD$; then there must be some rule in $\mathcal{P}$ (namely $A \longrightarrow BCD$) such that the added variable(s) (B in this case) is nullable. So we add a child node with label B to the node with label A and append a sub-tree of yield ϵ rooted at B. This is now a parse tree for G with yield w.



Additional Proofs

Outline of Proof of Theorem 7.1.4

$\underline{L(G_{\text{no-unit}}) \subseteq L(G)}$: By definition, $A \to \gamma$ in $P_{\text{no-unit}}$ iff there exists a $B \in V$ such that $A \overset{*}{\underset{G}{\Rightarrow}} B$ and $B \longrightarrow \gamma$ in $\mathcal{P}$.

- Thus, every production rule $A \to \gamma$ of $P_{\text{no-unit}}$ is effectively a derivation $A \overset{*}{\underset{G}{\Rightarrow}} \alpha$ in G.
- Hence, every derivation of $G_{\text{no-unit}}$ is a derivation of G.

$\underline{L(G) \subseteq L(G_{\text{no-unit}})}$: Consider a derivation of $w \in L(G)$ from S.

- Argue that every run of unit productions in $\mathcal{P}$ that are used in this derivation must be followed by a non-unit production rule in $\mathcal{P}$.
- Each such run of unit productions in $\mathcal{P}$ followed by a non-unit production can be condensed to a single production in $P_{\text{no-unit}}$. [See definition of $P_{\text{no-unit}}$]
- The condensed derivation is then a derivation of w using rules in $P_{\text{no-unit}}$.

Additional Proofs

Proof of Theorem 7.1.7

(1) $L(G_{\text{GR}}) \subseteq L(G))$ since the alphabets and the rule of G_{GR} are subsets of those of G.

- Suppose $w \in L(G)$. Then, there must be such a derivation of w from S:

$$\underset{\text{Rule: }}{}S \underset{G}{\Rightarrow} \gamma_1 \underset{G}{\Rightarrow} \gamma_2 \underset{G}{\Rightarrow} \gamma_3 \cdots \underset{G}{\Rightarrow} \gamma_k = w.$$

Rule: R_1 R_2 R_3 R_k

- Since every variable symbol that appears in this derivation is generating, they and the production rules $R_1, \ldots, R_k$ used in this derivation will be present in G_{G}.
- Every variable in this derivation is reachable; consequently, the variables that appear and the rules $R_1, \ldots, R_k$ will be present in G_{GR}. Then, $w \in L(G_{\text{GR}})$.

(2) A straightforward exercise in verifying the definition on Slide 7. Note that the remaining symbols have to be shown to be useful in G_{GR}, and not in G!

Additional Proofs

Outline of Proof of Theorem 7.1.8

- $L(G) \subseteq L(\hat{G})$ because every production rule of $\hat{G}$ has a corresponding equivalent derivation of α from A in $\hat{G}$.
- Consider the parse tree of $w \in L(\hat{G})$. If there are no introduced variables, then this is also the parse tree of w in G and hence $w \in L(G)$.
- If there are introduced variables, replace them by the complex production in G that introduced them in the first place (such replacements are always possible). The resultant tree is a parse tree for w in G, and hence $w \in G$.