COMP3630 / COMP6363

week 5: **Introduction to Turing Machines**

This Lecture Covers Chapter 8 of HMU: Introduction to Turing Machines

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The Australian National University

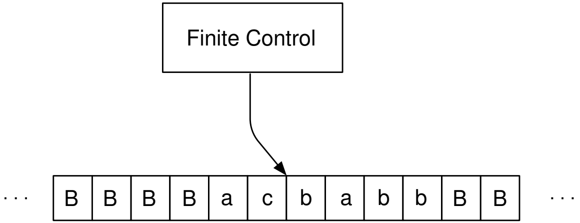
Semester 1, 2023

Content of this Chapter

- Turing Machine
- Extensions of Turing Machines (and PDAs)
- Restrictions of Turing Machines

Additional Reading: Chapter 8 of HMU.

Turing Machine: Informal Definition



- An tape extending infinitely in both sides
- A reading head that can edit tape, move right or left.
- A finite control.
- A string is accepted if finite control reaches a final/accepting state

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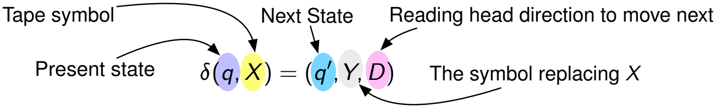
- δ: (deterministic) transition function. δ is a **partial function** over $Q \times \Gamma$, where the first component is viewed as the present state, and the second is viewed as the tape symbol read. If $\delta(q, X)$ is defined, then



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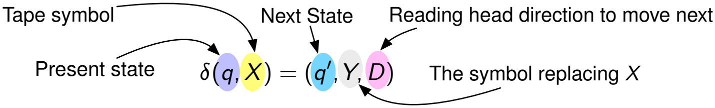

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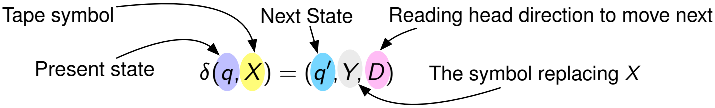

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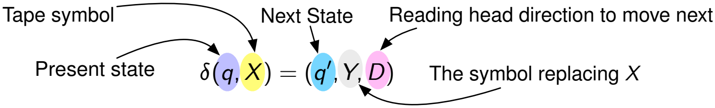

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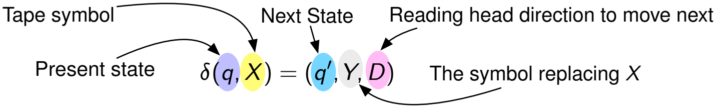

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- Head **always** moves to the left or right. Being stationary is not an option. It can also be defined with such an option, see tutorial.

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- Turing machines can be defined by describing δ using a transition table.
- They can also be defined using transition diagrams (with labels appropriately altered)

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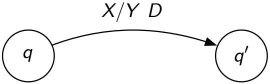

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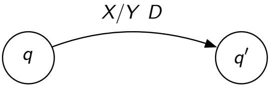

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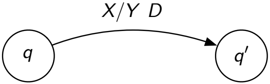

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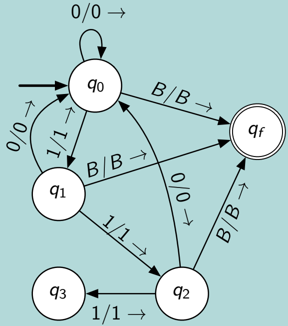

This encodes a DFA (almost).
Can you see why?

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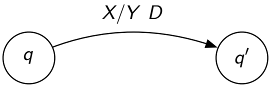

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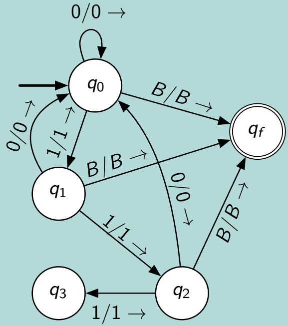

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Because we never manipulate the tape and terminate once the String is read. The only difference is that not all edges are defined, but this can be fixed with a trap state.

Instantaneous Descriptions of TMs

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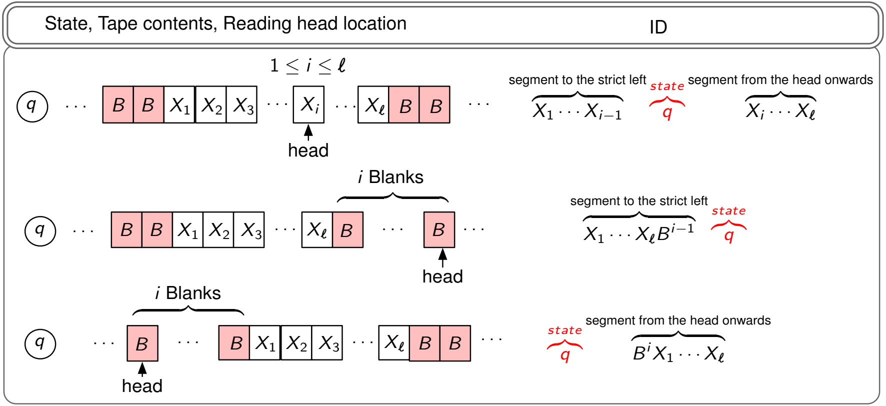
- The instantaneous description or configuration or ID of a TM contains 3 parts:
  (a) The (finite, non-trivial) portion of tape to the left of the reading head;
  (b) the state that the TM is presently in; and
  (c) the (finite, non-trivial) portion of the tape to the right of the reading head.



'Moves' of a TM

- Just as in the case of a PDA, we use $\underset{M}{\vdash}$ to indicate a single move of a TM M, and $\overset{*}{\underset{M}{\vdash}}$ to indicate zero or a finite number of moves of a TM.

Present ID	Transition	**Next** ID
$X_1 \cdots X_{i-1} q X_i \cdots X_\ell$	$\delta(q, X_i) = (q', Y, R)$	$X_1 \cdots X_{i-1} Y q' X_{i+1} \cdots X_\ell$
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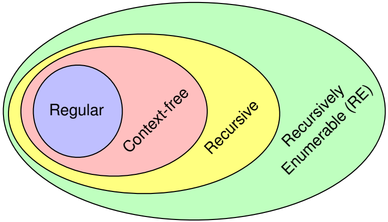

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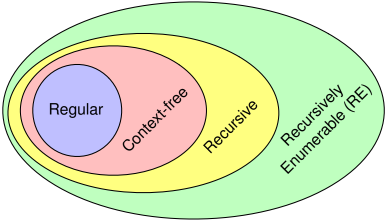

- (Another important class is the context-sensitive languages. They sit between the context-free and recursive languages.)

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  - However, if you have to judge properties of a given TM (e.g., whether some TM M always halts, or accepts a particular word etc.), then you have to deal with *any* TM – reasonable or not... (*Motivation:* You might want to judge properties of somebody's "program". And we'd like to know whether we actually can!)

On Acceptance, Rejection, Termination, and Deciding II/III

- “Accepting w”

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On Acceptance, Rejection, Termination, and Deciding III/III

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- Note that $R \subsetneq RE$, and therefore being in RE, does not mean that we are not in R! We might be since for any $L \in R$ holds $L \in RE$.

Multiple-Track TMs

Multiple-track TM

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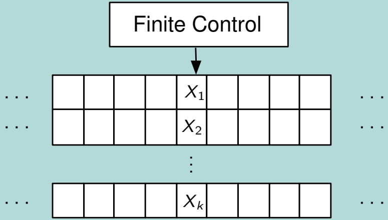
- There are k tracks, each having symbols written on them. They are essentially tapes, but we call them that way since they are not independent.



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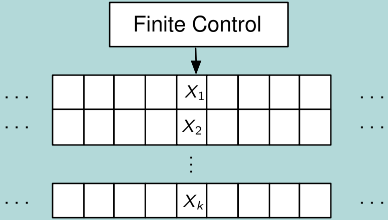

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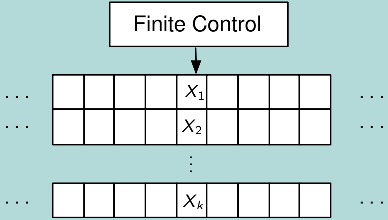

- A k-track TM with tape alphabet Γ has the same langauge-acceptance power as a TM with tape alphabet Γ^k. (E.g., each cell contains the “symbol” $(X_1, \ldots, X_k)$)

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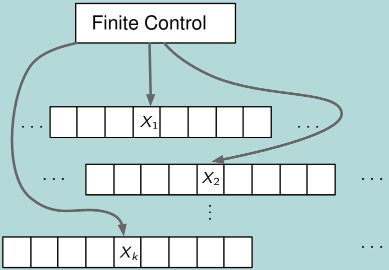

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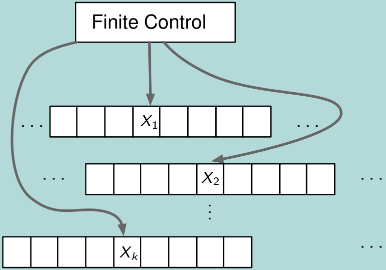

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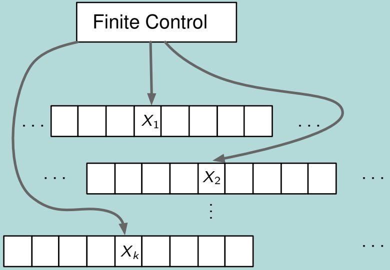

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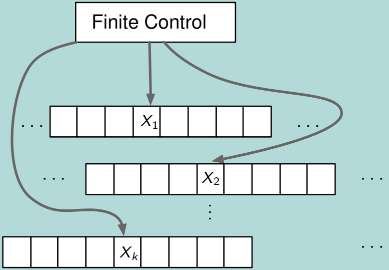

- The rest stays the same (e.g., one set of states, acceptance, etc.).

Multi-tape TMs

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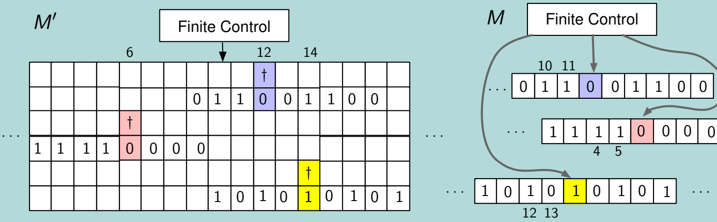

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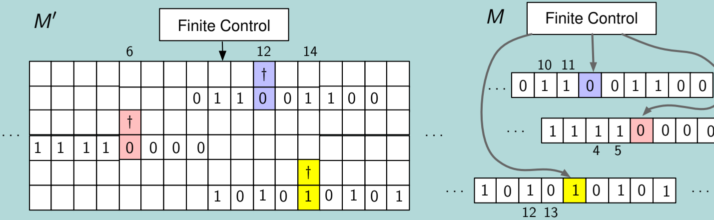
  The $2i-1^{\text{th}}$ track of M' contains $\dagger$ at the location where the i^{th} head of M is located.

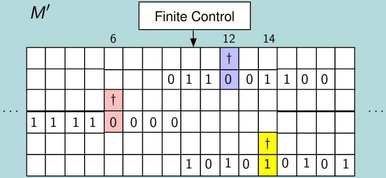


Multi-tape TMs

Proof of Theorem 8.2.1 (1 of 3)

What is the main problem we need to solve?

- In the Multi-tape TM M, heads move independently, whereas in the Multi-track TM M' they do not. So the heads can diverge:



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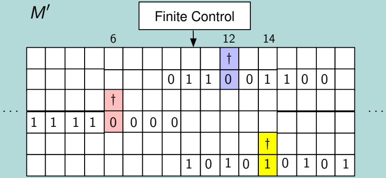

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So, how to solve it?

- Make sure that in each transition of M, we visit all heads of M'.
- "Store" all head positions in a state with k (number of tapes) entries.

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Proof of Theorem 8.2.1 (2 of 3)

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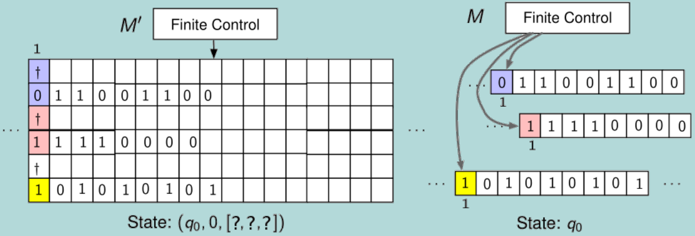

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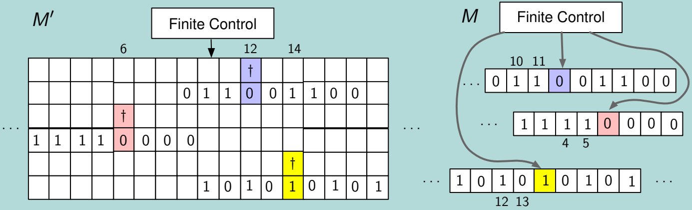
- Each move of M takes multiple moves of M', and is a sweep of the tape from the location of the leftmost † to that of the rightmost † and back performing the changes to tracks that M would do to its corresponding tapes.
- The right sweep ends when the second component is k.



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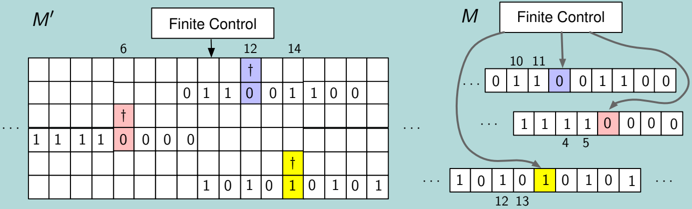

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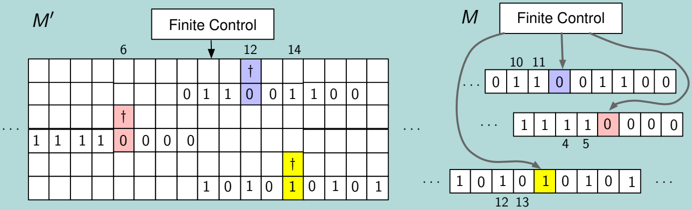

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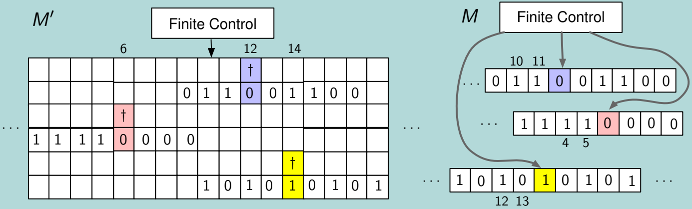

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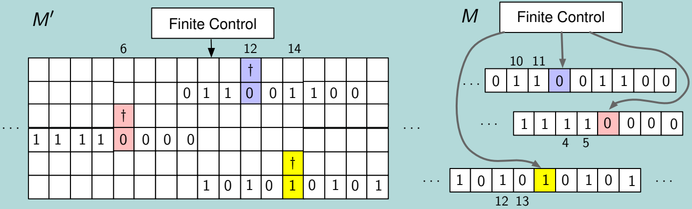

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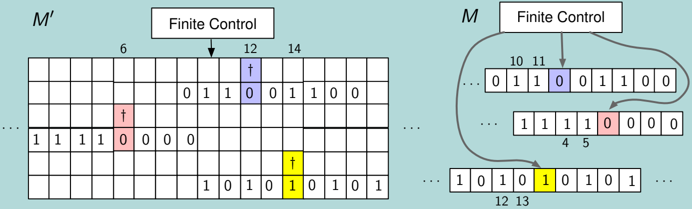
- Note that M' mimics M and hence the languages accepted are identical.



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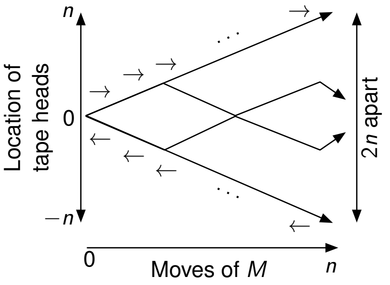

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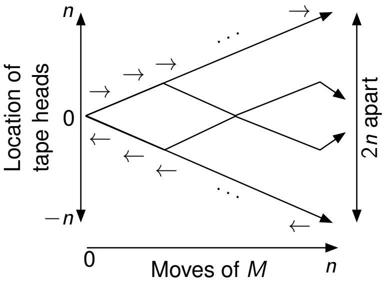

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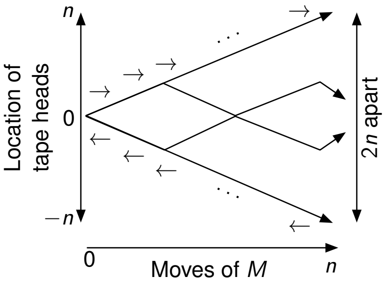
- Each track update requires $\Theta(k)$ time.



Multi-tape TMs

- The running time of a TM M with input w is the number of moves M makes before it halts. (If it does not, the running time is ∞).
- The time complexity $T_M : \{0, 1, \ldots\} \to \{0, 1, \ldots\} \cup \{\infty\}$ of a TM M is defined as follows:
  - $T_M(n) :=$ maximum running time of M for an input w of length n symbols.

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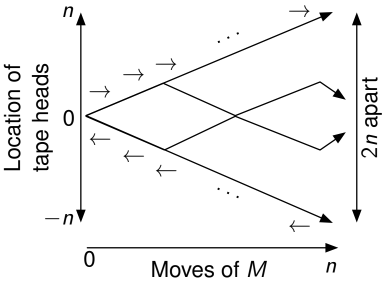
- Each track update requires $\Theta(k)$ time.
- So n moves in M need $O(n^2)$ moves in M'.



Non-deterministic TMs

Non-deterministic TM: $\delta(q, X)$ is a set of triples representing possible moves.

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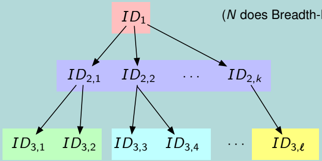

(N does Breadth-First exploration of IDs of M)

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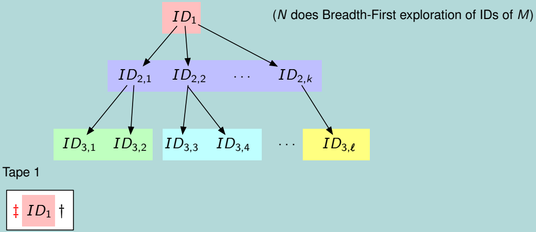

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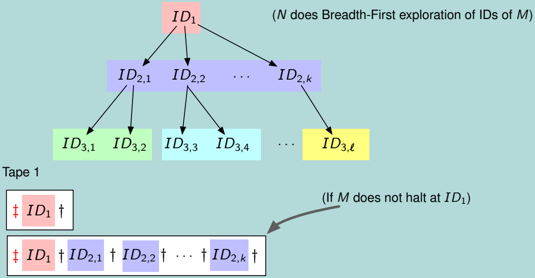

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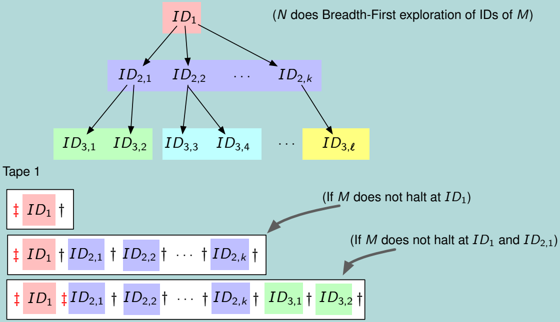

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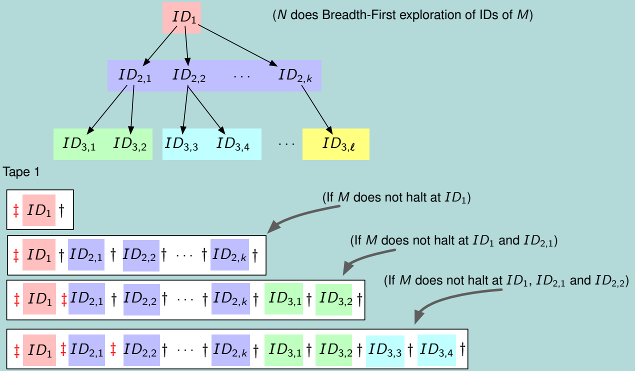
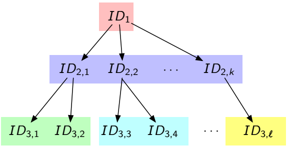
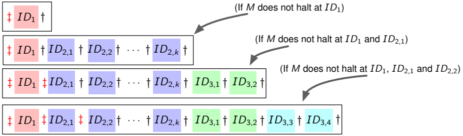


Outline of Proof of Theorem 8.2.3

- We can devise a 2-tape TM N that simulates M.
- N first replaces the content of the first tape by ‡ followed by the ID that M is initially in, which is then followed by a special symbol †, which serves as ID separator. (N uses the second tape as scratch tape to enable this operation).
- If the ID corresponds to a final state, N accepts (as would M).
- If not, N then identifies all possible choices for the next IDs for M and enters each one of them followed by † at the right end of its first tape. (Again, N uses the second tape as scratch tape to enable this operation.)
- N then searches for † to the right of ‡, changes the † to a ‡ (to signify that it is processing the succeeding ID), and processes that ID in the similar way.
- N halts at an ID iff M would at that ID.





TM Semi-infinite Tape

A TM with a semi-infinite tape is a TM that only has blanks on one of its sides, but not on the other.

Phrased (slightly) more formally:
A TM with a semi-infinite tape is a TM that can never move to left of the left-most input symbol.

We don't provide a formal definition, but a way of simulating this is by providing a special symbol, placed on the left of the input, and defining the transitions to always go to the right when this is read.

TM Semi-infinite Tape

Theorem 8.3.1

Every recursively enumerable language is also accepted by a TM with semi-infinite tape.

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- Given a TM M that accepts a language L, construct a two-track TM M' as follows.
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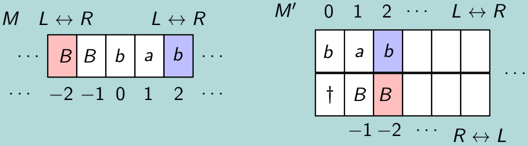

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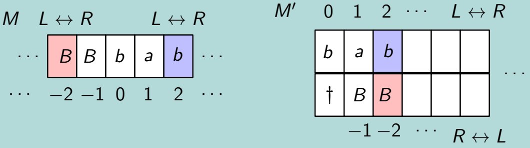

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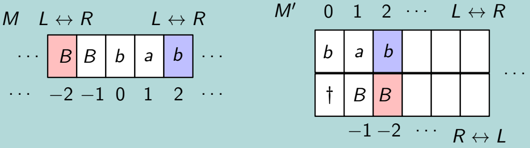

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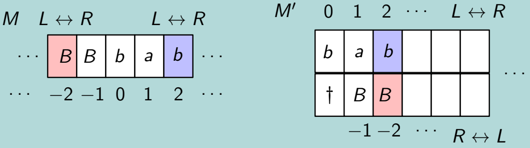
- At any time, M' keeps track of whether M is to the right or left of its start location.
- If M is to the strict right of its start location, M' mimics M on the first track. If M is to the strict left of its start location, M' mimics M on second track, but with the head directions reversed. M' detects the start by the † symbol.



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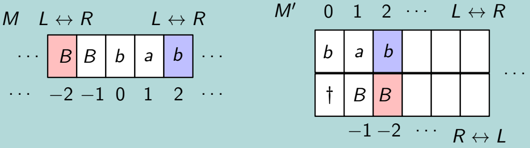
- It can be formally shown that M' accepts a string iff M accepts it.



Multi-stack Machines

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Theorem 8.4.1

Every recursively enumerable language is accepted by a two-stack PDA

Outline of Proof of Theorem 8.4.1

- Let each stack again contain a bottom-most start symbol.
- Let ID $= x_{-3}x_{-2}x_{-1}qx_0x_1x_2$, i.e., $w = x_{-3}x_{-2}x_{-1}x_0x_1x_2$, and head read reads x_0
- Let stack-1 be $x_0x_1x_2$ (top to bottom) the head and its right and stack-2 be $x_{-1}x_{-2}x_{-3}$ the head's left part in reversed order.
- What if we move the head to the right? Then, ID' $= x_{-3}x_{-2}x_{-1}x_0q'x_1x_2$. We can easily do this with our stacks:
  - How should the stack now look like?
  - stack-1: x_1x_2 and stack-2: $x_0x_{-1}x_{-2}x_{-3}$.
  - But that's just a simple pop and push!
- Moving to the left, and changing the symbol that's written can be simulated as well.

Multi-stack Machines

Outline of Proof of Theorem 8.4.1, cont'd

- Remaining problem: How to fill the stacks initially?
- Recall: stack-1 contains the head and its right and stack-2 the head's left part in reversed order.

Multi-stack Machines

Outline of Proof of Theorem 8.4.1, cont'd

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Multi-stack Machines

Outline of Proof of Theorem 8.4.1, cont'd

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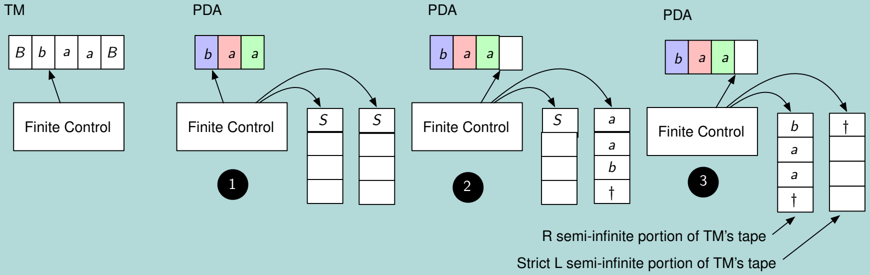
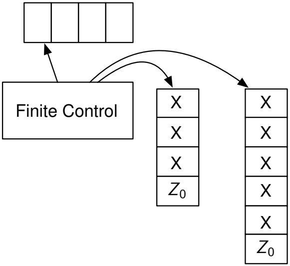
- Recall: stack-1 contains the head and its right and stack-2 the head's left part in reversed order.
- Initial configuration is $q_0 w$, so stack-1 should be w and stack-2 "empty".
- We achieve this by the following procedure:



- I.e., run to the right filling stack-2, then run back putting it on stack-1.

Counter Machines

- A counter machine is a multi-stack machine whose stack alphabet contains two symbols: Z_0 (stack end marker) and X
- Z_0 is initially in the stack.
- Z_0 may be replaced by $X^i Z_0$ for some $i \geq 0$
- X may be replaced by X^i for some $i \geq 0$.



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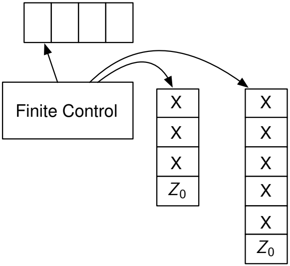
- A counter machine effectively stores a non-negative number.



Counter Machines

Theorem 8.4.2

Every recursively enumerable language is accepted by a three-counter machine

Outline of Proof of Theorem 8.4.2

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- WLOG, let the stack alphabet of $\Gamma = \{0, 1, \ldots, r-1\}$.
- Suppose stack 1/2 contains Y_1(top)$, \ldots, Y_k$. Then counter stores $Y_1 + rY_2 + \cdots + r^{k-1}Y_k$. E.g., if stack is $1, 5, 7$, interpret it as 157.

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- The third counter is used to change the two stack contents.
- Popping the top symbol from a stack (say A) $=$ finding quotient when $Y_1 + rY_2 + \cdots + r^{k-1}Y_k$ is divided by r.
  - pop r X's from stack A, and push a single X on the third stack. Repeat until all Xs are exhausted on the stack where popping is performed.
  - Now empty stack A and copy the third stack contents onto stack A.
- Change Y_1 to some Y_1' requires adding or subtracting, which is done by popping or pushing the corresponding number of Xs.

Counter Machines

Outline of Proof of Theorem 8.4.2

- pushing a symbol Z onto a stack (say A) $=$ compute $rC + Z$ where C is the number presently stored in the stack A.
  - pop one X from stack A, and push r Xs on the third stack.
  - Finally push Z Xs onto the third stack. Now empty stack A and copy the third stack contents onto stack A.
- Since the above three are the only operations needed to simulate a TM on a two-stack PDA, we can stimulate a 2-stack PDA and hence a TM using a 3-counter machine.

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Theorem 8.4.3

Every recursively enumerable language is accepted by a two-counter machine

Outline of Proof of Theorem 8.4.3

- The key idea: simulate three counters using one, and use the other for manipulations.
- The first counter stores $2^i 3^j 5^k$ where i, j, k are the contents of the 3-counter machine.
- Updates to the stack involve: (a) divide by 2, 3, or 5; (b) multiply by 2, 3, or 5; or (c) identify if i or j or k is zero (check divisibility).
- Each operation can be easily seen to be done with a spare counter.