COMP3630 / COMP6363

week 6: **Decidability and Undecidability**

This Lecture Covers Chapter 9 of HMU: Decidability and Undecidability

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convenor & lecturer: Pascal Bercher

The Australian National University

Semester 1, 2023

Content of this Chapter

- Preliminary Ideas
- Example of a non-RE language
- Recursive languages
- Universal Language
- Reductions of Problems
- Rice's Theorem
- Post's Correspondence Problem
- Undecidable Problems about CFGs

Additional Reading: Chapter 9 of HMU.

Enumeration of (Binary) Strings

- We can construct a bijective map ϕ from the set of binary strings $\{0,1\}^*$ to natural numbers $\mathbb{N}$.
  - Why might that appear surprising?
  - Because each number has a unique binary encoding, but for each we could add an arbitrary number of zeros in the front, so there seem to be more strings over $\{0,1\}$ than numbers in $\mathbb{N}$.

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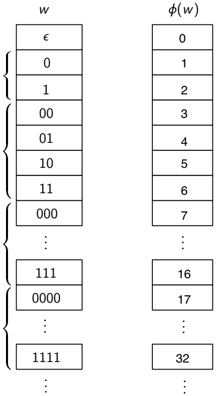
- Enlist all strings ordered by length, and for each length, order using lexicographic ordering.



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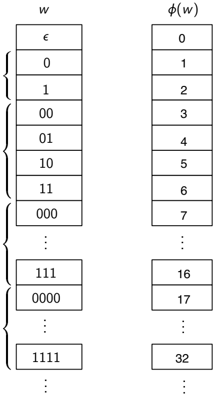
- Enlist all strings ordered by length, and for each length, order using lexicographic ordering.
- The set of finite binary strings is countable/denumerable.



A Code for Turing Machines

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- Consider $M = (Q, \Sigma = \{0, 1\}, \Gamma = \{0, 1, B, X_4, \ldots, X_\ell\}, \delta, q_1, B, F)$.
  - Rename states $\{q_1, \ldots, q_k\}$ for $k = |Q|$ with q_1: start state and q_k: final state.
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  - Rename L as D_1 and R and D_2. (The directions.)

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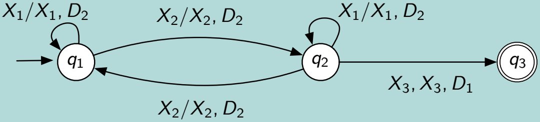
- Map each transition tuple (i,j,k,l,m) to a **unique** binary string $0^i10^j10^k10^l10^m$.
  NB: No string representing a transition tuple contains 11.
- Order transition tuples lexicographically and concatenate all transitions using 11 to indicate end of a transition. Let the resultant string be w_M. For example, 3 transitions can be combined as $\underbrace{0^{i_1}10^{j_1}10^{k_1}10^{l_1}10^{m_1}}_{\text{1st transition}}\,11\,\underbrace{0^{i_2}10^{j_2}10^{k_2}10^{l_2}10^{m_2}}_{\text{2nd transition}}\,11\,\underbrace{0^{i_3}10^{j_3}10^{k_3}10^{l_3}10^{m_3}}_{\text{3rd transition}}$
- For each TM M, define the code $\langle M\rangle$ for TM M as w_M.

The Set of Turing Machines

An Example: A TM that accepts strings with odd # of 1s



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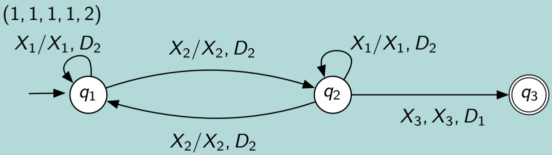

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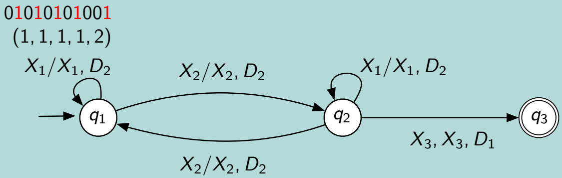

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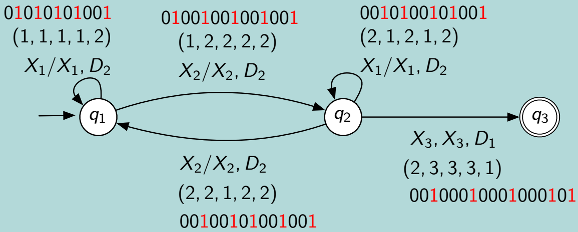

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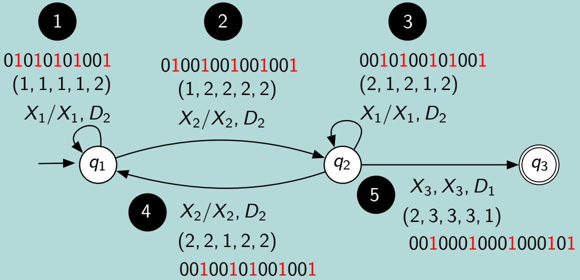

$$< M >= 010101010^2 111010^2 10^2 10^2 10^2 1110^2 1010^2 1010^2 111$$

$$0^2 10^2 1010^2 10^2 1110^2 10^3 10^3 10^3 101.$$

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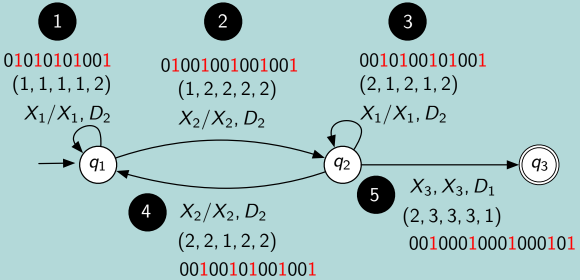

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$$0^210^21010^210^21110^210^310^310^3101.$$

Remark 9.1.1

- *Each TM M encoding has a unique natural number, i.e., $\phi(\langle M \rangle)$; Each TM M may have several codes $\langle M \rangle$ and thus several numbers; but each natural number corresponds to* **at most** *one TM.*

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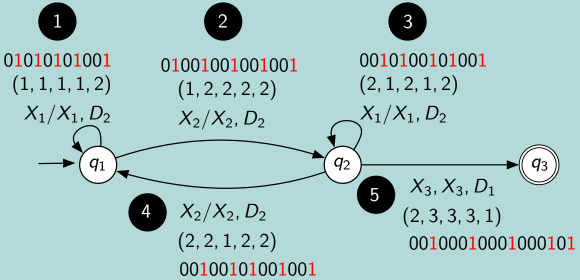

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- *Each TM M encoding has a unique natural number, i.e., $\phi(\langle M \rangle)$; Each TM M may have several codes $\langle M \rangle$ and thus several numbers; but each natural number corresponds to* **at most** *one TM.*
- *The set of TMs/RE languages/CFLs/regular languages is countable.*

Diagonalization Language L_d

- Let M_i be the TM s.t. $\phi(< M_i >) = i$. (If for an i, no such TM exists, we let M_i to be the TM with 1 state, no transitions and no final state, i.e., it accepts no input).
- Construct an infinite table. Rows: M_0, M_1, ... as above and cols: All Strings according to slide 3. Cell $(i, j) = 1$ iff M_i accepts $w_j := \phi^{-1}(j)$.
- Define a language $L_d = \{w_j : M_j \text{ does not accept } w_j, \text{ where } j \in \mathbb{N}\}$.

	✔ ϵ $\phi^{-1}(0)$	0 $\phi^{-1}(1)$	1 $\phi^{-1}(2)$	✔ 00 $\phi^{-1}(3)$	01 $\phi^{-1}(4)$	✔ 10 $\phi^{-1}(5)$	11 $\phi^{-1}(6)$	...
M_0	0 ✔	0	0	0	0	0	0	
M_1	1	1	0	0	0	1	1	
M_2	0	1	1	1 Text	0	0	1	
M_3	1	1	1	0 ✔	0	1	1	
M_4	1	0	0	1	1	0	0	
M_5	1	1	0	0	0	0 ✔	1	
⋮								

$L_d = \{\epsilon, 00, 10, \ldots\}$

† Entries are for illustrative purposes only

L_d is not recursively enumerable language

- L_d cannot be accepted by **any** TM.

	✔			✔		✔		
	ϵ	0	1	00	01	10	11	...
	$\phi^{-1}(0)$	$\phi^{-1}(1)$	$\phi^{-1}(2)$	$\phi^{-1}(3)$	$\phi^{-1}(4)$	$\phi^{-1}(5)$	$\phi^{-1}(6)$	
M_0	**0** ✔	**0**	**0**	**0**	**0**	**0**	**0**	
M_1	**1**	**1**	**0**	**0**	**0**	**1**	**1**	
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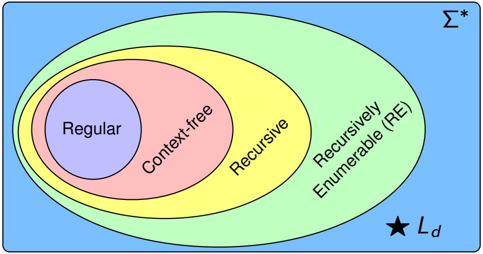
Recursive Languages

- A language L is **recursive** if it is accepted by a TM M that halts on **all** inputs
  - In such a case, the TM M is said to **decide** L.
  - Every recursive language is recursively enumerable (by definition).



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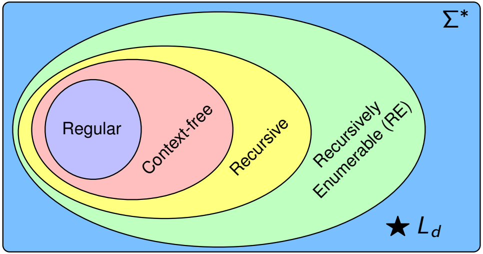

- Do not confuse deciding with accepting! TMs can accept without always terminating (namely, e.g, for languages in $RE \setminus R$, where R denotes the recursive languages).

(Some Obvious) Properties of Recursive Languages

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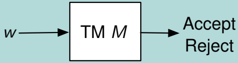
If L is **recursive***, so is L^c.*

Proof of Theorem 9.3.1



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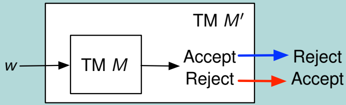

- Accepting states of M with $L(M) = L$ are non-accepting states of M' with $L(M') = L^c$.
- Add a new and only final state q_f in M' such that:

$$\delta_M(q, X) \text{ undefined and } q \notin F$$
$$\Downarrow$$
$$\delta_{M'}(q, X) = (q_f, X, R).$$

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(Some Obvious) Properties of Recursive Languages

Theorem 9.3.2

If L and L^c are both recursively enumerable, then L (and L^c) are recursive.

Proof of Theorem 9.3.2

- Let $L = L(M)$ and $L^c = L(M')$. Run M and M' in parallel using a 2-tape TM.
- Both TMs cannot halt in final states, and both TMs cannot halt in non-final states.
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Alternate Definition of Recursive Languages

L is recursive if both L and L^c are recursively enumerable.

The Universal Language and Turing Machine

Universal Language L_u

- $L_u := \{\langle M \rangle 111 w : \text{ TM } M \text{ and } w \in L(M)\}$. [See Slide 3]

Universal TM U (modelled as 5-tape TM)

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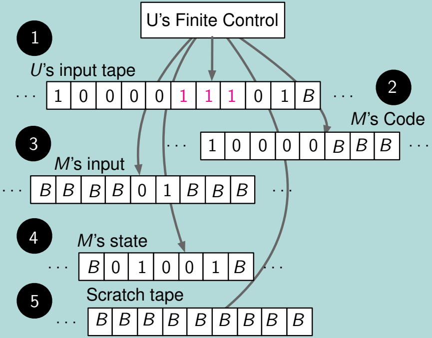

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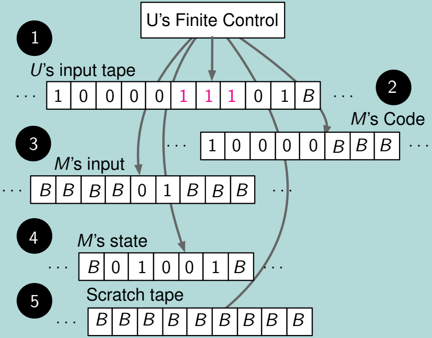

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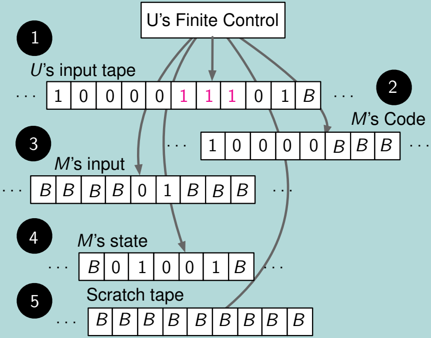

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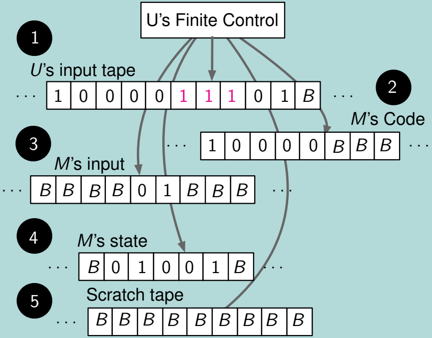
3. Initiates 4th tape with 0^1 (M starts in q_1)
4. To simulate a move of M, U reads tapes 3 and 4 to identify M's state and input as 0^i and 0^j; if state is accepting, M (and hence U) accepts its inputs and halts. Else, U scans tape 2 for 110^i10^j1 or $BB0^i10^j1$.
   - If found, using the transition, tapes 4 and 3 are updated, and tape 3's head moves to right or left.
   - If not, M halts, and so does U.



Where does L_u Lie in the Hierarchy of Languages?

Theorem 9.4.1

L_u is recursively enumerable, but is not recursive.

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- Let TM M' accept $w \in L_u^c$ and reject $w \in L_u$.

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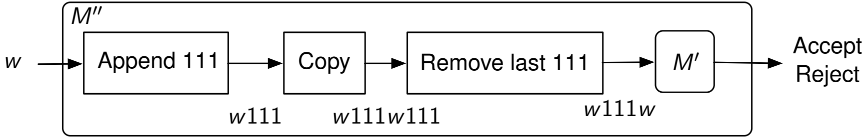

Where does L_u Lie in the Hierarchy of Languages?

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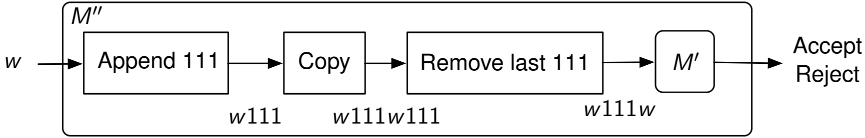

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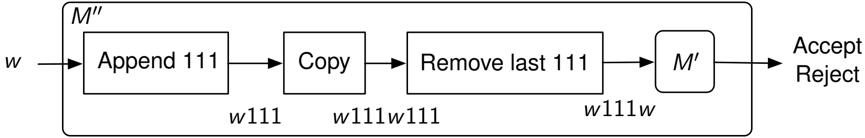

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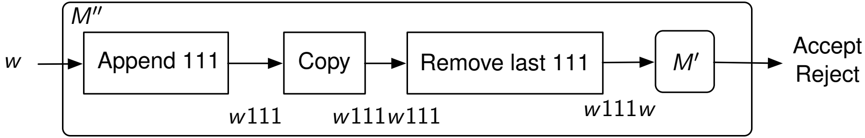

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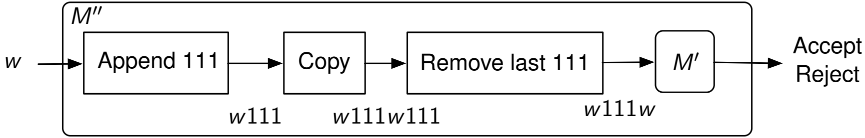
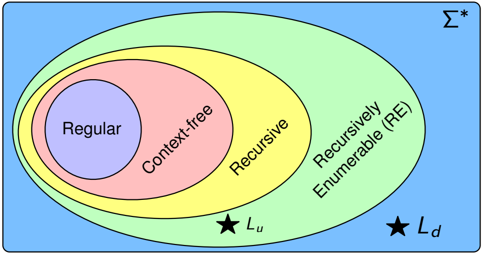
- M'' accepts $w \iff w111w \in L_u^c \iff w111w \notin L_u \iff w \in L_d$.
- Then, $L(M'')$ **is** the diagonal language L_d, which is impossible!



Recap

- There exists a bijection $\phi : \Sigma^* \to \mathbb{N}$.
- There exists an injective function $< \cdot >$: Set of TMs $\to \Sigma^*$.
- RE languages are countable.



- The diagonalization Language L_d is not recursively enumerable.
- Recursive languages are closed under complementation. (See tutorials for more!)
- The universal language $L_u = \{\langle M \rangle 111w : M \text{ accepts } w\}$ is RE, but not recursive.

What is a Reduction?

- A decision problem P is said to reduce to decision problem Q if **every** instance of P can be transformed to **some** instance of Q and a yes (or no) answer to that instance of Q yields a yes (or no) answer to original instance of P, respectively.
  - We did already make use of reductions in this lecture multiple times!
  - E.g., reduce the problem of deciding L^c to the problem of deciding L: Here the new problem was only a minimal modification, by flipping results (see slide 9).
- Here, **transform** implies the existence of a Turing machine that takes an instance of P written on a tape and **always halts** with an instance of Q written on it.

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Theorem 9.6.1

If a problem P reduces to a problem Q then:

(a) P is undecidable $\Rightarrow$ Q is undecidable.

(b) P is non-RE $\Rightarrow$ Q is non-RE.

Problem Reduction

Proof of Theorem 9.6.1

(a) P **is undecidable** $\Rightarrow$ Q **is undecidable**.
Suppose P is undecidable and Q is decidable. Let TM M_Q decide Q.

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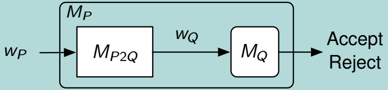

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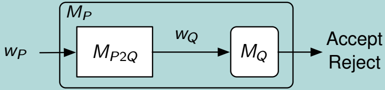

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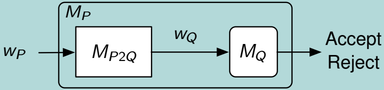

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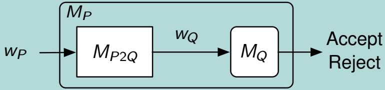

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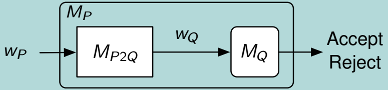

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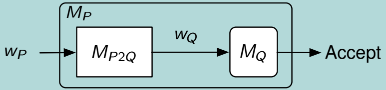

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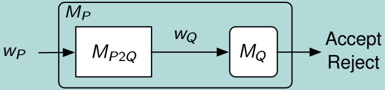

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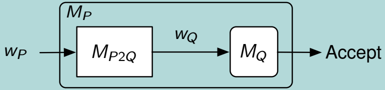

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Some More Abstract Languages

Language of TMs Accepting Empty and Non-empty Languages

- $L_e = \{\langle M \rangle : L(M) = \emptyset\}$.
- $L_{ne} = \{\langle M \rangle : L(M) \neq \emptyset\}$. (Note: $L_{ne} \neq L_e^c$, because some strings don't encode TMs.)

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Theorem 9.7.1

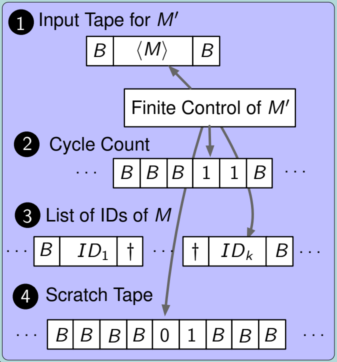
L_{ne} is RE.

Note that this theorem doesn't say whether it's recursive or not!

L_{ne} is RE.

Proof of Theorem 9.7.1 (using "dovetailing")

- In cycle k, M' runs one move of M for each ID, and adds the initial ID of M when $\phi^{-1}(k)$ is on the tape.
- ID(i,j) = the ID after $j-1$ moves when M reads $\phi^{-1}(j)$ on its tape.
- If any ID contains an accepting state, M' halts as M would have on that input.



Cycle	Tape 1	Tape 2
1	1	$ID(1,1)$
2	10	$ID(1,2)$ † $ID(2,1)$
3	11	$ID(1,3)$ † $ID(2,2)$ † $ID(3,1)$
$\vdots$	$\vdots$	$\vdots$
k	$101\cdots0$	$ID(1,k)$ † $ID(2,k-1)$ † $ID(3,k-2)$ † $\cdots$ † $ID(k,1)$

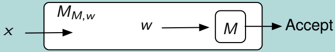
L_{ne} is not recursive

Theorem 9.7.2

L_{ne} is not recursive.

Proof of Theorem 9.7.2

- For every TM M and string w, there is a TM $M_{M,w}$ that ignores its input and runs M on w: $M_{M,w}$ erases its input tape, pastes w, and runs it as/on M.



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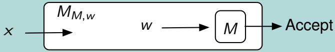

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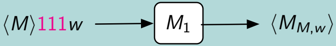

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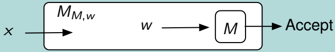

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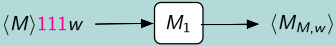

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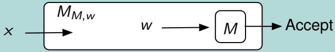

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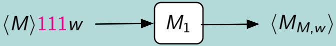

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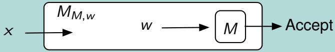

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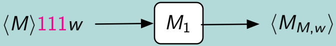

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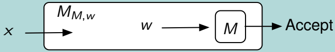

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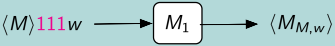

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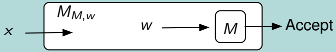

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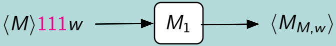

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- L_u is then recursive, which is a contradiction.

Rice's Theorem

Given: alphabet Σ and let $RE = \{L \subseteq \Sigma^* \mid L \text{ recursively enumerable}\}$.

- Recursively enumerable (RE) languages L corresponds to TM M if $L = L(M)$
- A **property** of RE languages is subset $\mathcal{P} \subseteq RE$ of the set of RE languages over Σ. Why do we call sets of languages a property? Think of examples:
  - $\mathcal{P}_1 = \{L \subseteq \Sigma^* : |L| < \infty\}$ (the property is being finite)
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- A property $\mathcal{P}$ is **trivial** if $\mathcal{P} = \emptyset$ or $\mathcal{P} = RE$ (and non-trivial otherwise).
- A property $\mathcal{P} \subseteq RE$ is decidable if $L_{\mathcal{P}} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}$ is decidable.

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  - $\mathcal{P}_2 = \{L \subseteq \Sigma^* : \text{there is a DFA D, s.t. } L = L(D)\}$ (the property is being regular)
- A property $\mathcal{P}$ is **trivial** if $\mathcal{P} = \emptyset$ or $\mathcal{P} = RE$ (and non-trivial otherwise).
- A property $\mathcal{P} \subseteq RE$ is decidable if $L_{\mathcal{P}} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}$ is decidable.

Theorem 9.7.3

*Every **non-trivial** property $\mathcal{P}$ of RE languages is undecidable, i.e., $L_{\mathcal{P}}$ is not recursive.*

Rice's Theorem

Given: alphabet Σ and let $RE = \{L \subseteq \Sigma^* \mid L \text{ recursively enumerable}\}$.

- Recursively enumerable (RE) languages L corresponds to TM M if $L = L(M)$
- A **property** of RE languages is subset $\mathcal{P} \subseteq RE$ of the set of RE languages over Σ.
  Why do we call sets of languages a property? Think of examples:
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- So Rice's theorem says something about some (many!) subsets
  $S \subseteq \{\langle M \rangle : \text{M is a TM}\}$ (So we want to know something about TMs!)

Rice's Theorem (Example 1)

How about the "property" that a TM has 10 states? (Should be decidable!)

- Let $L_{10} = \{\langle M \rangle : \text{M has 10 states}\}$. But we have to be able to write it as: $L_{10} = \{\langle M \rangle : L(M) \in \mathcal{P}\}$ where $\mathcal{P} \subseteq RE$ and not trivial.

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- So how about $\mathcal{P}_{10} = \{L \subseteq \Sigma^* : \text{there is a TM M, s.t. } L = L(M) \text{ and } M \text{ has 10 states}\}$?
- This doesn't work since we can take some M_9 with 9 states (and thus $\langle M_9 \rangle \notin L_{10}$) and add a dummy state, so we have 10 in the resulting TM M_{10}. Now we have:
  - $\langle M_9 \rangle \notin L_{10}$, and $\langle M_{10} \rangle \in L_{10}$, but
  - $L(M_9) = L(M_{10})$, so $L(M_9) \in \mathcal{P}_{10}$ and $L(M_{10}) \in \mathcal{P}_{10}$.
  - Recall $L_{\mathcal{P}} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}$, so $\langle M_9 \rangle \in L_{\mathcal{P}_{10}}$. ↯

  → So it doesn't work! It's not a property of languages! (So Rice's theorem doesn't apply.)

Rice's Theorem (Example 2)

How about the property that the language contains String "01"?

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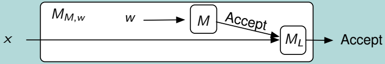
- Thus, $L_{\mathcal{P}_{01}} = \{\langle M \rangle : L(M) \in \mathcal{P}_{01}\}$ is undecidable. In other words: We can't decide whether a given TM accepts a language that contains a 01.

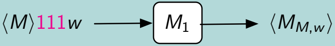
Rice's Theorem (Proof)

Proof of Theorem 9.7.3

- WLOG, we can assume that $\emptyset \notin \mathcal{P}$. Else consider $\mathcal{P}^c$.
- Since $\mathcal{P}$ is non-trivial, there is a language $L \in \mathcal{P}$ and a TM M_L that accepts L
- Let $M_{M,w}$ be a TM that runs M on w and if M accepts w, then reads its input and operates as M_L.



- **Mind-bending step:** There is a TM M_1 that takes $\langle M \rangle 111 w$ and outputs $\langle M_{M,w} \rangle$. Note: M_1 **always** halts (even if M does not halt when input is w!)



- M accepts $w \iff L(M_{M,w}) = L \in \mathcal{P}$
- If $\mathcal{P}$ were decidable, then there is a TM M_2 such that M_2 accepts $\langle M \rangle$ iff $L(M) \in \mathcal{P}$.
- Then, we can devise a TM M_3 such that it reads $\langle M \rangle 111 w$ operates first as M_1 and then when M_1 has halted, it operates as M_2.
- M_3 accepts/rejects $\langle M \rangle 111 w \iff L(M_{M,w}) \in / \notin \mathcal{P} \iff M$ accepts/rejects w.
- Then, L_u is recursive, a contradiction

PCP: Definition

- Suppose we are given two ordered lists of strings over Σ, say $A = (u_1, \ldots, u_k)$ and $B = (v_1, \ldots, v_k)$. We say (u_i, v_i) to be a **corresponding pair**.
- PCP Problem: Is there a sequence of integers $i_1, \ldots, i_m$ such that:

  $$u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_m}?$$

  - m can be greater than the list length k.
  - We can reuse pairs as many times as we like.

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  $u_{i_1} \cdots u_{i_m}$
  $= v_{i_1} \cdots v_{i_m}$?
  - m can be greater than the list length k.
  - We can reuse pairs as many times as we like.

A PCP example

A	110	0011	0110
B	110110	00	110

- A solution cannot start with $i_1 = 3$.
- A solution can start with $i_1 = 1$, but then $i_2 = 1$, and $i_3 = 1 \ldots$ Consequently, i_1 cannot equal 1.
- A solution does exist: $(i_1, i_2, i_3) = (2, 3, 1)$.
- $(i_1, i_2, i_3, i_4, i_5, i_6) = (2, 3, 1, 2, 3, 1)$ is also a solution.

Modified PCP (MPCP): Definition

- Suppose we are again given two ordered lists of strings over Σ, say $A = (u_1, \ldots, u_k)$ and $B = (v_1, \ldots, v_k)$.
- MPCP Problem: Is there a sequence of integers $i_1, \ldots, i_m$ such that

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- The previous example does not have a solution when viewed as an MPCP problem.
- So MPCP is indeed a different problem to PCP, but...

Theorem 9.8.1

MPCP reduces to PCP

MPCP: Thoughts/Ideas before constructing a Proof

- So we want to prove that MPCP reduces to PCP.
- More specifically we need to:
  - Turn every MPCP problem into a PCP problem (with preserving solutions).
  - I.e., **how can we enforce PCP to always select the first element first?**

Thus, the problem we need to solve is:

- To make sure that that the first string gets selected first, but
- without making additional solutions available or cutting some out!

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Thus, the problem we need to solve is:

- To make sure that that the first string gets selected first, but
- without making additional solutions available or cutting some out!

Initial thoughts:

- We add a new start symbol to u_1 and v_1 so that they match.
- But that still doesn't enforce that we start with them! ...

Outline of Proof of Theorem 9.8.1

- Given MPCP's lists $A = (u_1, \ldots, u_k)$ and $B = (v_1, \ldots, v_k)$. We now transform this into a PCP problem! Suppose that symbols $\diamond, \triangle$ are not in the strings of A and B.

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  - For $i = 1, \ldots, k$,
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A	B
110	110110
0011	00
0110	110

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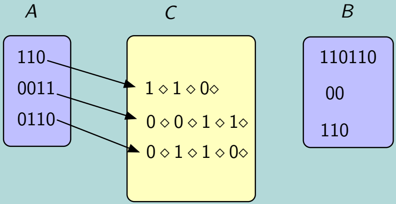

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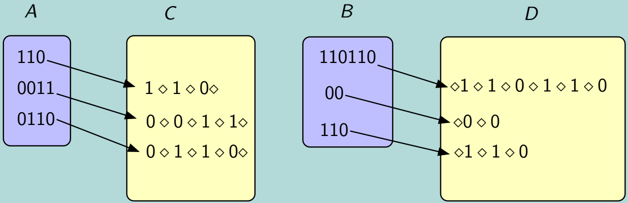

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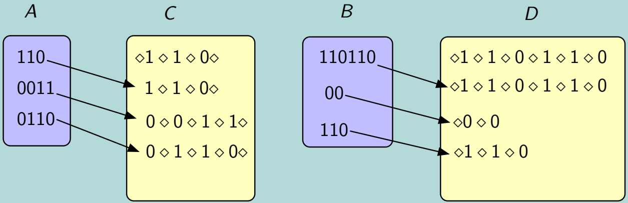

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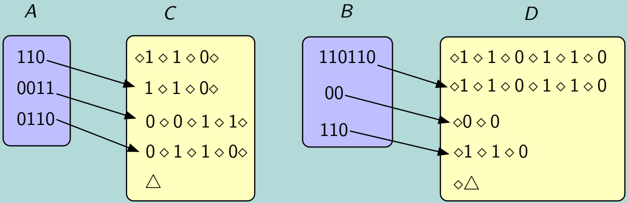

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PCP is undecidable

Theorem 9.8.2

PCP is undecidable.

Outline of Proof of Theorem 9.8.2 (Overview)

We reduce L_u to MPCP (and did already MPCP to PCP).

PCP is undecidable

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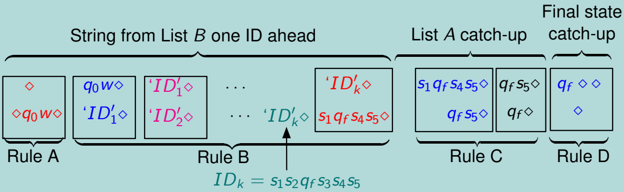
- Hence, MPCP is undecidable. [following Theorem 9.6.1]
- Since MPCP is undecidable, PCP is also undecidable. [following Theorem 9.6.1]

So the hard work is to solve/model $\langle M \rangle 111w \in L_u$ via MPCP!

PCP is undecidable (More detailed proof at the end)

Outline of Proof of Theorem 9.8.2 (Overview)

Abstract overview of existing pairs in the constructed MPCP:



The rules $A \ldots, D$ are in the appendix.

The overall idea is as follows:

- We have two lines of strings (which should match in the end).

PCP is undecidable (More detailed proof at the end)

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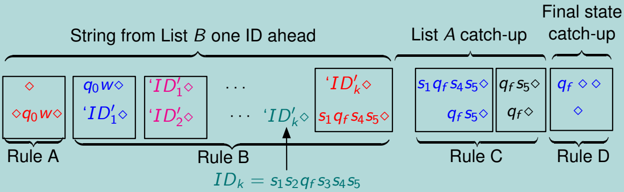

The rules $A \ldots, D$ are in the appendix.

The overall idea is as follows:

- We have two lines of strings (which should match in the end).
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PCP is undecidable (More detailed proof at the end)

Outline of Proof of Theorem 9.8.2 (Overview)

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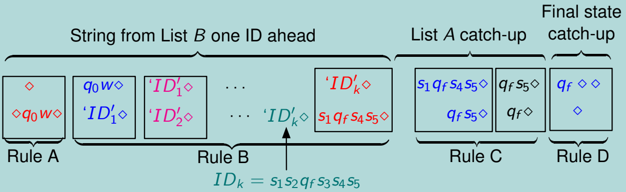

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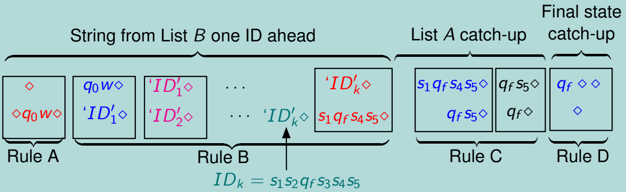

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- We construct a pair for every valid TM transition! (Rule B) In such a pair, the first line/entry is the old configuaration and the second the new.
- We have/need a few more rules to make all strings equal and deal with final states. Note how we have to move the first line to get matchings strings. (Rules C, D)

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Proof of Theorem 9.8.2 (Short Example)

Before we look at an example, recap:

- A TM ID looks as: $X_1 \ldots, X_{i-1} q X_i \ldots X_\ell$ where X_i is below the head.

Now, with TM's start state q_0 and initial tape $w = s_1 s_2 s_3$ let:

- Word in line 1: $\diamond$
- Word in line 2: $\diamond q_0 s_1 s_2 s_3 \diamond$

We get this by our first pair, created by Rule A:

- First entry in 1st list: $\diamond$
- First entry in 2nd list: $\diamond q_0 s_1 s_2 s_3 \diamond$

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What's next? Create the transitions! (Via Rules in B)

- Assume $\delta(q_0, s_1) = (p, t_1, R)$, then $q_0 s_1 s_2 s_3 \underset{M}{\vdash}$
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We get this by another pair, created by Rule B:

- Entry in 1st list: $q_0 s_1$ since $\delta(q_0, s_1) = (p, t_1, R)$
- Entry in 2nd list: $t_1 p$ and thus $q_0 s_1 s_2 s_3 \underset{M}{\vdash} t_1 p s_2 s_3$

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What's next? The remaining symbols from last configuration are missing...

- We add a pair (s, s) for all $s \in \Gamma$ (Rule I)
- and one pair $(\diamond, \diamond)$ (Rule I)

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We get this by several new pairs, created by Rule I:

- $(s_0, s_0), (s_1, s_1), (s_2, s_2), \ldots$ (for all $s \in \Gamma$)
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What's next? We continue! Next transition!

- Assume $\delta(p, s_2) = (r, t_2, L)$, then $t_1 p s_2 s_3 \underset{M}{\vdash}$
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We get this by another pair, created by Rule B:

- Entry in 1st list: $t_1 p s_2$ — since $\delta(p, s_2) = (r, t_2, L)$
- Entry in 2nd list: $r t_1 t_2$ — and thus $t_1 p s_2 s_3 \underset{M}{\vdash} r t_1 t_2 s_3$

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What's next?

- First, we again add the missing symbols, until
- eventually we find a final state. We have more rules for that (see appendix).

› We'll now revisit CFGs and prove that ambiguity in CFGs is undecidable.

Theorem 9.9.1

The problem if a CFG is ambiguous is undecidable.

Outline of Proof of Theorem 9.8.2

› We'll reduce ... which one? (1) PCP to CFG or (2) CFG to PCP?

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Theorem 9.9.1

The problem if a CFG is ambiguous is undecidable.

Outline of Proof of Theorem 9.8.2

- We'll reduce every instance of a PCP problem to a CFG.
- Given a PCP problem with $A = (w_1, \ldots, w_k)$ and $B = (x_1, \ldots, x_k)$, pick symbols $a_1, \ldots, a_k$ that don't appear in any string in list A or B.
- Now define a grammar G with production rules

$$
\begin{aligned}
S &\longrightarrow A \mid B \\
A &\longrightarrow w_1 A a_1 \mid \cdots \mid w_k A a_k \mid w_1 a_1 \mid \cdots \mid w_k a_k \\
B &\longrightarrow x_1 B a_1 \mid \cdots \mid x_k B a_k \mid x_1 a_1 \mid \cdots \mid x_k a_k
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B &\longrightarrow x_1 B a_1 \mid \cdots \mid x_k B a_k \mid x_1 a_1 \mid \cdots \mid x_k a_k
\end{aligned}
$$

- If there are two leftmost derivations of a string in $L(G)$, one must use $S \longrightarrow A$ and $S \longrightarrow B$, respectively.
- Every solution to the PCP leads to 2 leftmost derivations of some string in $L(G)$ and vice versa. (Note how the solution indices are encoded in the end of each word.)
- Since PCP is undecidable, the ambiguity of CFGs must be undecidable [Thm 9.6.1]

Overview of (Some) Undecidable Problems Concerning CFGs

- Given a CFG G, is it ambiguous? (We just had that.)

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- Given a CFG G, is it ambiguous? (We just had that.)
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  (As mentioned before, this is used to show that HTN planning is undeciable)

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- Given CFG G and regular language L, is $L(G) = L$?
- Given CFG G and regular language L, is $L \subseteq L(G)$?
- Given CFG G, is $L(G) = \Sigma^*$?

PCP is undecidable

Proof Details of Theorem 9.8.2 (Rule Definitions)

- For the proof we construct an MPCP for each TM M and input w.

Rule A: Construct two lists A and B whose first entries are $\diamond$ and $\diamond q_0 w \diamond$, respectively.

Rule I: Add corresponding pairs (X, X) (for all $X \in \Gamma$) and $(\diamond, \diamond)$

Rule B: Suppose q is not a final state. Then, append to the list the following entries:

List A	List B	
qX	Yp	$if\, \delta(q, X) = (p, Y, R)$
ZqX	pZY	$if\, \delta(q, X) = (p, Y, L)$
$q\diamond$	$Yp\diamond$	$if\, \delta(q, B) = (p, Y, R)$
$Zq\diamond$	$pZY\diamond$	$if\, \delta(q, B) = (p, Y, L)$

Rule C: For $q \in F$, let (XqY, q), (Xq, q), and (qY, Y) be corresponding pairs for $X, Y \in \Gamma$

Rule D: For $q \in F$ $(q \diamond \diamond, \diamond)$ is a corresponding pair.

PCP is undecidable

Proof Details of Theorem 9.8.2 (Construction/Explanation)

- Suppose there is a solution to the MPCP problem. The solution starts with the first corresponding pair, and the string constructed from List B is already an ID of TM M ahead of the string from List A.
- As we select strings from List A (corresponding to Rule B) to match the last ID, the string from List B adds to its string another valid ID.
- The sequence of IDs constructed are valid sequences of IDs for M starting from q_0w.
- Suppose the last ID constructed in the string constructed from List B corresponds to a final state, then we can gobble up one neighboring symbol at a time using Rule C.
- Once we are done gobbling up all tape symbols, the string from List B is still one final state symbol ahead of List A's string.
- We then use Rule D to match and complete.