COMP1600, week 12:

Problem Complexities

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slides based on those by: Dirk Pattinson

(with contributions by Victor Rivera and previous colleagues)

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Australian National University

Overview of Week 12

- Motivation
- Non-Deterministic Turing Machines
- Big-$\mathcal{O}$ Notation
- Complexity Classes
- Reductions
- NP-Completeness

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  - Which problems are equally hard? Which ones are harder than others?
  - We look at how problems can be "turned into each other".
- We measure "performance" in terms of a Turing Machine's:
  - Time requirement (number of operations/transitions)
  - Space requirement (number of cells that can be read/written)

Why should we care for Problem Complexities?

Given a new problem to solve, we:

- ... can use existing solvers instead of designing new ones,
  - → *Which software do you think is better? The one you design from scratch in a few weeks, or one that entire research communities (few or dozens to thousands of PhD students, post-docs, Professors) created over decades?*

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- ... understand the problems we solve much better.
  - → *If you know that your (new) problem is equivalent to an existing (established) one, that surely helps... (Imagine, you take a course twice! The second time it's much easier...)*

Example (for the Relevance of this)

Boolean Satisfiability (SAT)

- Let ϕ be a boolean formula with n variables, e.g.,:
  $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_4) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_4)$ with $n = 4$.
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  - → SAT is in both **EXPTIME** and in **NP**, but **NP** is lower!
    Any idea *why* **NP** is lower? The guessing can be compiled away in exponential time! (Just try all options.)
- But is the problem also in **P**? (I.e., can we **P**-solve it without guessing?)

Non-Deterministic Turing Machines

Deterministic Turing Machines, Recap

A **Turing Machine** has the form $(S, s_0, F, \Gamma, \Sigma, B, \delta)$, where

- S is the set of **states** with $s_0 \in S$ the **initial state**;
- $F \subseteq S$ are the **final states**;
- Γ is the set of **tape symbols** (everything that might ever be on the tape);
- $B \in \Gamma \setminus \Sigma$ is the **blank symbol**;
- $\Sigma \subseteq \Gamma$ is the set of **input symbols**;
- δ is a (partial) **transition function**

$$\begin{array}{rcl} \delta \ : \ S \times \Gamma & \rightharpoonup & S \times \Gamma \times \{L, R, S\} \\ (\text{state}, \text{tape symbol}) & \mapsto & (\text{new state}, \text{new tape symbol}, \text{direction}) \end{array}$$

The direction tells the read/write head which way to go next:
Left, Right, or Stay/Stop. (Stopping the head is different from halting.)

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(We could also have formalized this with a transition relation.) The direction tells the read/write head which way to go next: Left, Right, or Stay/Stop. (Stopping the head is different from halting.)

Fundamental Properties of Non-Det. TMs

Basic Properties/Definitions

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- The language of a Non-deterministic Turing Machine is the words accepted by it. (Just like for deterministic TMs.)
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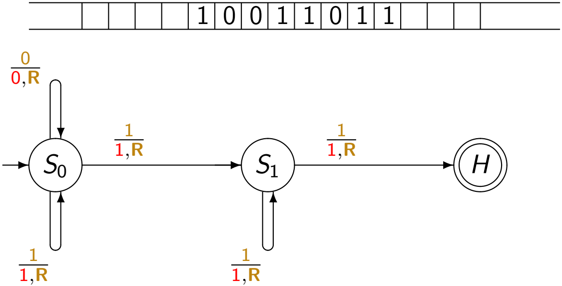
Relationship to Deterministic TMs.

- Non-det. TMs can't do *more* than deterministic ones.
- Non-det. TMs could be *quicker* than deterministic ones. (Unknown!)

Example (for a Non-Det. TM)

Consider the following TM, defined over an initial string over $\Sigma = \{0, 1\}$:



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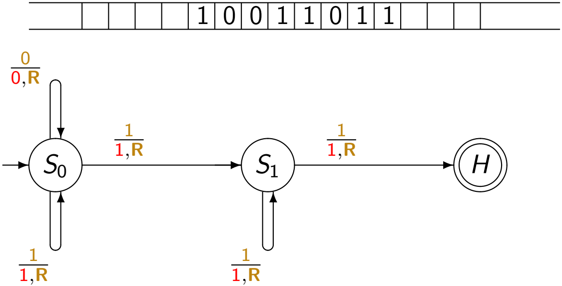

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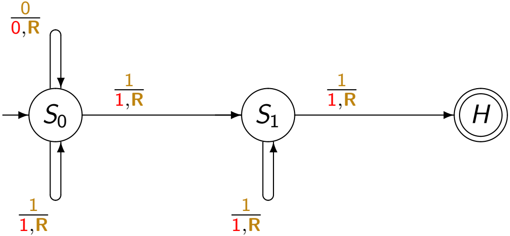

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Checking for the "right" input.

$\{w \mid w \text{ contains} \geq 2 \text{ consecutive 1s}\}$
$= \{w11w' \mid w, w' \in \Sigma^*\}$

Big-$\mathcal{O}$ Notation

Introduction

- In the following we will define complexity classes based on whether some TM with specific properties exists.
- Example SAT: There is a non-det. TM that runs in "polynomial time" – what does that mean?
- We formalize this using the Big-$\mathcal{O}$ notation.
- That way we will know whether some function (the runtime or space consumption of a TM) is in polytime or exponential time etc.

Poll. Who of you knows the big-$\mathcal{O}$ notation already?

Example

Let's decide $L = \{0^i 1^i \mid i \in \mathbb{N}\}$ by TM M, i.e., check whether an arbitrary input has the form $0^i 1^i$ for some i. M does:

- Scan word w and reject if 10 is found.
- Repeat as long as there are 0s and 1s on the tape:
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   f adheres $f(2k) = f(2(k-1)) + 4k + 1$, which is in $\mathcal{O}(n^2)$.

w	ϵ	01	$0^2 1^2$	$0^3 1^3$	$0^4 1^4$	$0^5 1^5$
$f(\|w\|)$	2	8	19	34	53	76

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So in total, M has polynomial time and space restriction!

Time Complexity – Abstraction

Problem. Exact "number of steps function" usually *very complicated*

- for example, $2n^{17} + 23n^2 - 5$
- and hard to find in the first place (see last slide!).

Solution. Consider *approximate* number of steps

- focus on *asymptotic* behaviour
- as we are only interested in *large* problems

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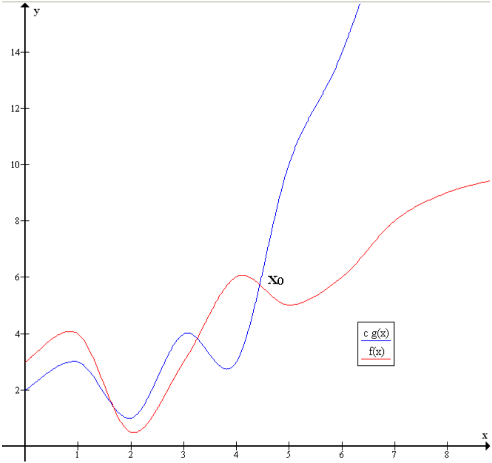
Big-$\mathcal{O}$ Notation. for f and g functions on natural numbers

- $f \in \mathcal{O}(g)$ if $\exists c.\ \exists n_0.\ \forall n \geq n_0.\ f(n) \leq c \cdot g(n)$
- "for large n, g is an upper bound to f up to a constant."
- E.g,. $f(n) \in \mathcal{O}(n^{17})$, since $g(n) = n^{17}$ and we can choose $c = 3$ so that we have $3n^{17} \geq f(n)$ for all $n \geq n_0$ (for a suitable n_0)

Graphical Illustration

Recall: $f \in \mathcal{O}(g)$ if $\exists c.\ \exists n_0.\ \forall n \geq n_0.\ f(n) \leq c \cdot g(n)$



Here, $f(x) \in \mathcal{O}(g(x))$ since $g(x)$ is at least as high as $f(x)$ for all $x \geq x_0$

Examples

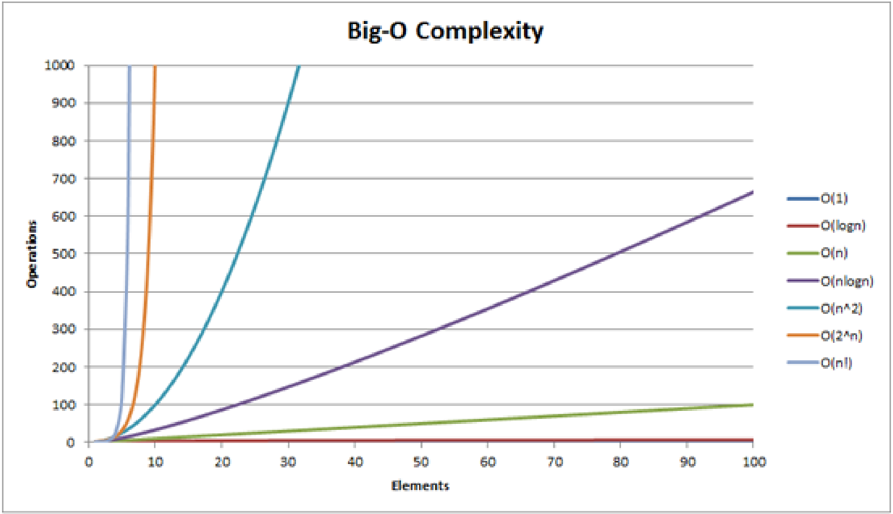
Examples.

- Polynomials: leading exponent dominates
- e.g. "$x^n +$ lower powers of x" $\in \mathcal{O}(x^n)$
- Exponentials: dominate polynomials
- e.g. "$2^n +$ polynomial" $\in \mathcal{O}(2^n)$

Important Special Cases.

- *linear*. f is linear if $f \in \mathcal{O}(n)$
- *polynomial*. f is polynomial if $f \in \mathcal{O}(n^k)$, for some k
- *exponential*. f is exponential if $f \in \mathcal{O}(2^n)$

Important Special Cases, Graphically



(Image copyright Lauren Kroner)

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We focus on classes **P** vs. **NP** vs. **EXPTIME**!
(The remaining ones are just listed for the sake of completeness.)

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For this course, only need to know about **P**, **NP**, and **EXPTIME**.

Note how every problem in **P** is also in **NP**. So if a problem is in **NP**, is it an *easy* one from **P** or a hard one like SAT? Hence: *Completeness!* (Later)

Reductions

Basic Definitions

This is the most important (and fun!) part of this week!

- We want to transform problems into each other – via *reduction*.
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$A \subseteq \Sigma_1^*$ is *polynomial time mapping-reducible* to $B \subseteq \Sigma_2^*$, written $A \leq_P B$, if a polytime-computable function $f : \Sigma_1^* \to \Sigma_2^*$ exists that is also a reduction (from A to B).

Reductions

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- A reduction is a polynomial-time translation of the problem, say r.
- More precisely:
  1. $r(w)$ can be computed in time polynomial in $|w|$.
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Example:

- $\text{EVEN} := \{n \mid n \bmod 2 = 0\}$, $\text{ODD} := \{n \mid n \bmod 2 = 1\}$
- Reduction from ODD to EVEN:
  - $r(k) = k + 1$, so we get $k \in \text{ODD}$ iff $r(k) \in \text{EVEN}$
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  - This shows that EVEN is at least as hard as ODD.
- If however our goal would have been to show that the 'new' problem ODD is at least as hard as EVEN, then we would have had to reduce from EVEN to ODD (though r would have been the same). Check this statement after "hardness" was introduced!

Example: Independent Set

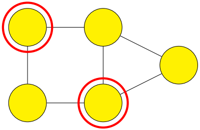
The Independent Set Problem:
Assume you want to throw a party. But you know that some of your friends don't get along. You only want to invite people that *do* get along.

Formalized as graph.

- vertices are your mates
- draw an edge between two vertices if people don't get along

Problem:
Given a graph and a $k \geq 0$, is there an *independent set*, i.e., a subset I of $\geq k$ vertices so that

- no two elements of I are connected with an edge.
- i.e., everybody in I gets along



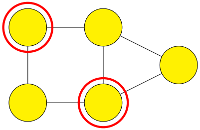
Example of an independent set of size 2 (*just* the red-circled vertices)

Solving the Independent Set Problem

Naive Implementation:

- loop through all subsets of size $\geq k$ (exponentially many!)
- and check whether they are independent sets

$\rightarrow$ Proves membership in **EXPTIME**



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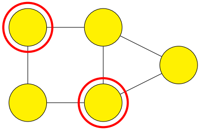

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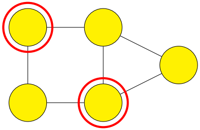

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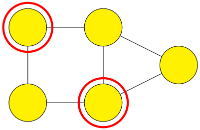

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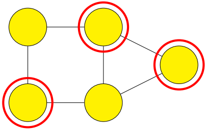
Answer: We don't know! But "hardness" helps giving a partial answer.

Example 2: Vertex Cover

Given a graph $G = \langle V, E \rangle$, a *vertex cover* is a set C of vertices such that every edge in G has at least one vertex in C.



Example vertex cover:
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- Given graph $\langle V, E \rangle$ and $k \geq 0$, is there a vertex cover of size $\leq k$?
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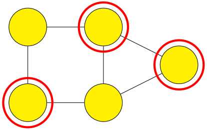

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Naive Algorithm:

- search through all subsets of size $\leq k$ (this is exponential)
- check whether it's a vertex cover

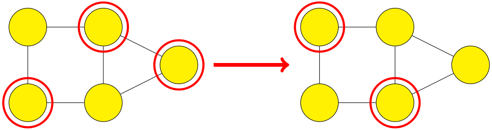
$\rightarrow$ This proves $VC \in$ **EXPTIME**, but we can do better!
(I.e., we could also guess and verify as before, giving $VC \in$ **NP**.)

From Independent Set to Vertex Cover

Reductions. Use solutions of one problem to solve another.

Observation. Let G be a graph with n vertices and $k \geq 0$.

- G has a VC of size $\leq k$ iff G has an IS of size $\geq n - k$



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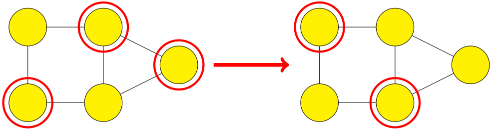

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What's the reduction? Vertex cover to independent set:

- $\langle G, k \rangle \in VC$ iff $r(\langle G, n \rangle) \in IS$, where $= r(\langle G, n \rangle) = \langle G, n - k \rangle$.
- Here, the reduction r only changes the number, but nothing else. But for most reductions, we will have to "translate problems", e.g., when turning a SAT problem into a VC problem (or vice versa)!

Important Note on Reductions

Be aware!

- So far, we only reduced problems, which were "equally hard", they were just "different flavors of the same problem":
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  - You should be able to reduce Vertex Cover to the Sokoban game. (Reducing an **NP**-complete problem to one that's **PSPACE**-hard.)

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Membership, Hardness, Completeness

Membership, Hardness, and Completeness

Definition (NP completeness, NP membership, NP hardness)

A language B is ***NP**-complete* if

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- Therefore, **NP**-complete problems are the hardest ones in **NP**. (In particular they may be harder than those in **P**!)
- Hardness is the opposite of "practical exploitation of reductions": For hardness, reduce *from* a known problem rather than *to* one!

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  - It doesn't even exist! (bail!)
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Why are we interested in showing **NP**-hardness/completeness in the first place?

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  - Since the problem could be even harder! (E.g., **PSPACE**-hard, **EXPTIME**-hard, **NEXPTIME**-hard, . . . , and *infinitely* more!)
  - Each problem class has specific "properties". E.g., "**NP**-complete looks like Logic", "**PSPACE**-complete looks like planning", etc.

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$\rightarrow$ It gives us a convenient procedure to show **NP**-completeness!

- First, show **NP** membership. (That's almost always very easy.)
- Then, show hardness by grabbing an **NP**-complete problem and reduce it to yours!

Known **NP**-Complete Problems

List of known **NP**-complete problems:

- SAT (first problem proved **NP**-hard) and 3-SAT (see tutorials)
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One of the most important problems in computer science is: $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Summary

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In weeks 7 to 11:

- We started with “machines” to recognizing only regular expressions.
- We added bits of computation power until we obtained a machine that can compute everything that’s possible. (Cf. Chosky Hierarchy.)
- On top languages from type 0 to 3, we also differentiate between recursive, recursively enumerable, and not recursively enumerable.

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Conclusion

Some concluding words:

- I hope you enjoyed the course, and especially weeks 7 to 12! :)
- COMP3630, Theory of Computation, equals weeks 7 to 12 maxed out, for example:
  - The equivalence of different TM models is proved there (e.g., multi-tape TMs or semi-infinite TMs)
  - When we had proof sketches, the course covers the proof. (E.g., it shows that SAT is **NP**-hard (and hence **NP**-complete.)
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- Good luck in the exam!