COMP1600, week 8: Non-Deterministic Finite Automata (NFAs) and Regular Expressions

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slides based on those by: Dirk Pattinson
(with contributions by Victor Rivera and previous colleagues)

Semester 2, 2024

Australian National University

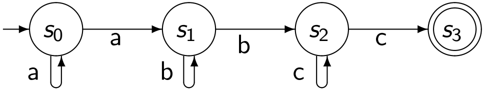
Overview of Week 8

- Introduction
- Non-Deterministic Finite Automata (NFAs) — Formally —
- Language of an NFA
- Determinisation of NFAs
- NFAs with ϵ-transitions
- Regular Expressions

Introduction

Non-Deterministic Finite State Automata — NFAs

Consider this FSA:



Q. Is it a DFA in the sense of our definition?

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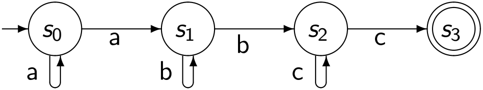

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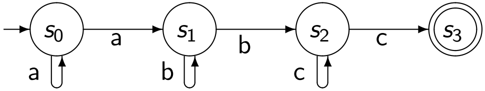
Q. Is it intuitively clear what it does? Test it! What's it's language?

Is it legal, i.e., a "proper" DFA?



A. It makes sense, but it is *nondeterministic*: A nondeterministic finite automaton (NFA). So not a "legal" DFA, but a specimen of a different breed.

Differences to deterministic automata

- Multiple edges with the **same label** come out of states
- For some states, there is **not an edge** for every token

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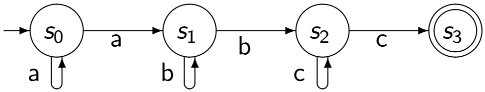

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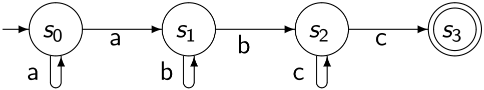
Formally. NFAs have a transition *relation* rather than a transition *function*.

- transition relation $R(s_1, x, s_2)$ is true if there's an x-labelled edge from s_1 to s_2
- there can be *many* states that are connected to s_1 via an x-labelled edge. (Example: s_0, s_1, s_2)
- there can be *no* x-labelled edge between s_1 and *any* state. (Example: s_3)

Is it clear what it does?



Observations.

- Some states don't have an outgoing edge with a certain letter, so the NFA can "get stuck".
- In some states, there's more than one possible successor state with a certain letter.

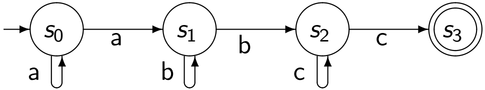
Acceptance condition for NFAs given string w:

- can get from initial to final state, making the "right" choice of successor state without getting stuck

Example. $w = aaabcc$

- need to "look ahead" to make the right choice
- (alternatively, try to backtrack if wrong choice has been made)

Is it clear what it does?



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Non-Example. $w = aaacc$

- Doesn't work because we are (definitely) stuck after reading the last a.

Key Differences: DFAs *vs* NFAs

DFA:

- DFAs have a *transition function*.

NFA:

- NFAs have a *transition* ***relation***.

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- An input sequence $x_1, x_2, \ldots, x_n$ is *accepted* by a DFA if there *exists* some sequence of transitions that leads from the initial state to a final state.

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- NFAs have a *transition* ***relation***.
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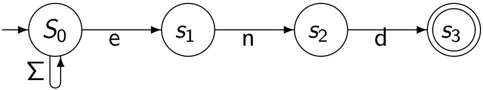
- An input sequence $x_1, x_2, \ldots, x_n$ is *accepted* by a NFA if there *exists* some sequence of transitions that leads from the initial state to a final state.

Q. Is there actually a difference between the solution criteria?

Example: NFA vs. DFA

$L = \{\alpha end \mid \alpha \in \Sigma^*\}$ An NFA recognising strings of letters ending in "end":
(The alphabet Σ here is the Latin alphabet.)



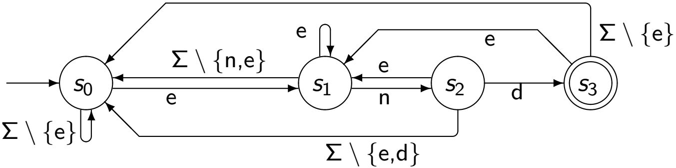
Note.

- *two* transitions from s_0 for the letter "e"
- *no* transition from s_1 for (e.g.) the letter "d"

An Equivalent DFA

Example. DFAs are (often) more complex.

A DFA that recognises strings of letters than end in “end”.



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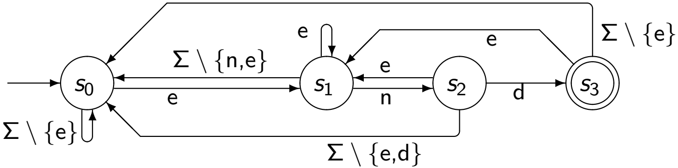
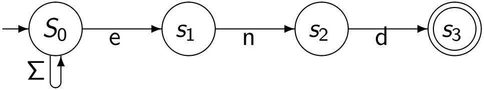


Q. Which FSA is easier to write and read?



Why do we need/use/have NFAs?

So, why do we have NFAs?

- They are more compact.
- They are (sometimes!) easier to read and write.

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Why do we need/use/have NFAs?

So, why do we have NFAs?

- They are more compact.
- They are (sometimes!) easier to read and write.
- Because we are step-wise increasing the power of our models of computation! (this week: add non-determinism.)

Q1. Why only sometimes?

Q2. Can you think of another reason why "NFAs exist"?

Non-Deterministic Finite Automata (NFAs) — Formally —

NFAs: Formal Definition

A Nondeterministic Finite State Automaton (NFA) consists of five parts:

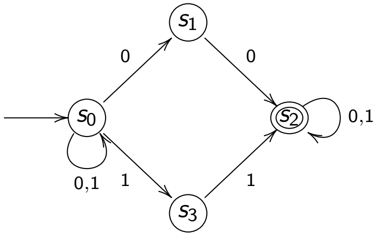
$$A = (\Sigma, S, s_0, F, R)$$

- a finite input **alphabet** Σ, the (finite) set of tokens
- a finite set of **states** S
- an **initial** state $s_0 \in S$ (we start here)
- a set of **final** states $F \subseteq S$ (we hope to finish in one of these)
- a **transition relation** $R \subseteq S \times \Sigma \times S$.

Aside. The transition *relation* is what makes the automaton nondeterministic. It can be seen as a function $\delta : S \times \Sigma \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the set of subsets (i.e., power set) of S. (Cf. slide 18 of last week!)

Another Example

Transition Diagram



As a transition table.

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$\to s_0$	$\{s_0, s_1\}$	$\{S_0, s_3\}$
s_1	$\{s_2\}$	$\emptyset$
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Both convey precisely the same information.

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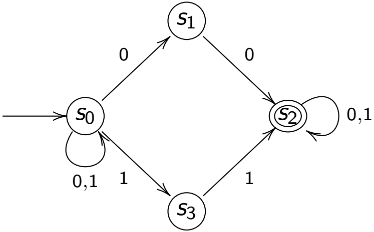

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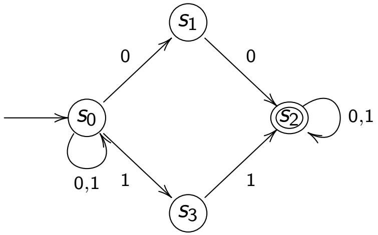

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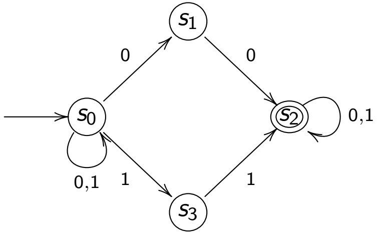

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  $L = (0 \mid 1)^*(00|11)(0 \mid 1)^*$ (this is a regular expression!)

Acceptance for NFAs

Acceptance Informally. An NFA $A = (\Sigma, S, F, s_0, R)$ *accepts* a word $w = x_1 x_2 \dots x_n$ (in symbols: $w \in L(A)$) iff there exists a sequence of states

$$s_0 \xrightarrow{x_1} s_1 \xrightarrow{x_2} \dots \xrightarrow{x_{n-1}} s_{n-1} \xrightarrow{x_n} s_n$$

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Aside. This is like for deterministic automata, the only difference is that for

- *deterministic automata* we have $s_i \xrightarrow{x} s_j$ if $N(s_i, x) = s_j$
  (that is, the automaton makes the (unique) transition)
- *non-deterministic automata* we have $s_i \xrightarrow{x} s_j$ if $(s_i, x, s_j) \in R$
  (that is, the automaton can make a transition)

Eventual State Relation for NFAs

Basic Idea. The eventual state relation $R^*(s, w, s')$ is true if s' is a state that the NFA ***can*** reach, starting in state s and reading string w.

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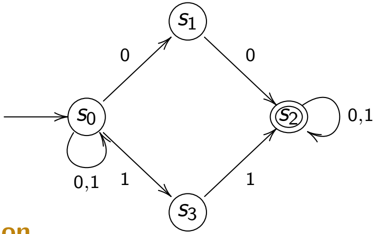
and is defined inductively as follows:

$$R^*(s, \epsilon, s) \text{ (is true)}$$
$$R^*(s, x\alpha, s') = \exists s''. R(s, x, s'') \wedge R^*(s'', \alpha, s')$$

Eventual State Relation: Example

The "double digits" automaton *DD*:



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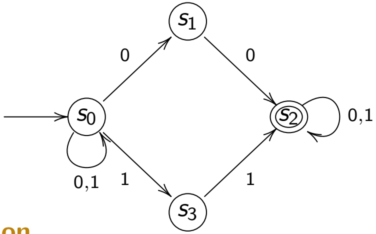

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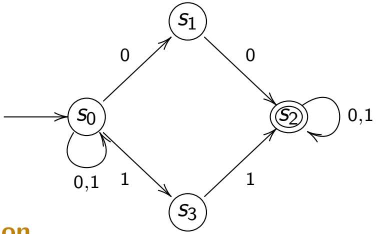

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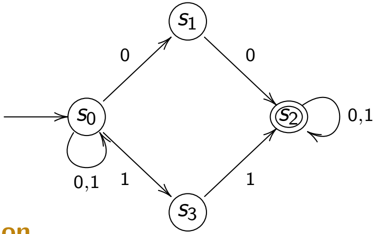

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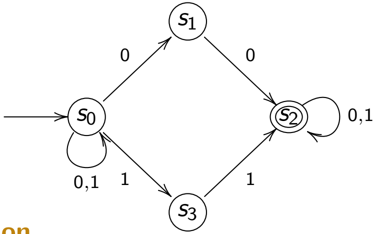

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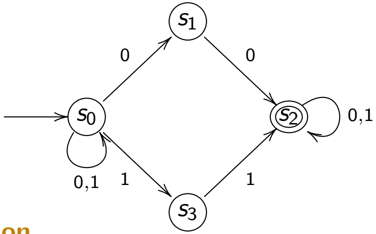

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Q2. Does $0110 \in L(DD)$ hold?

An Important (but Unsurprising) Theorem about R^*

For all states s, s' and for all strings $\alpha, \beta \in \Sigma^*$

$$R^*(s, \alpha\beta, s') \text{ if and only if } \exists s''.\ R^*(s, \alpha, s'') \land R^*(s'', \beta, s')$$

The proof is similar to the corresponding result for N^* in DFAs.
(You could do it as an exercise!)

Language of an NFA

Language of an NFA, revisited

Let $A = (\Sigma, S, s_0, F, R)$ be a NFA.

Acceptance, formally. A string w is *accepted* by A if

$$\exists s \in F.\ R^*(s_0, w, s)$$

(Compare with the definition of acceptance for NFAs earlier)

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(Compare with the definition of acceptance for NFAs earlier)

Language of an NFA.
The *language* accepted by A is the set of all strings accepted by A

$$L(A) = \{w \in \Sigma^* \mid \exists s \in F.\ R^*(s_0, w, s)\}$$

Informally. That is, $w \in L(A)$ iff *there exists* a path through the diagram for A, from s_0 to a final state s ($s \in F$), such that the symbols on the path match the symbols in w

Language of an NFA, Comment

Some comments (related to languages):

- Identifying the language of an NFA is not always easy!
- ... and neither is constructing an NFA given a language.
- We recommend practising:
  - Take some language and draw the NFA.
  - Take some NFA and identify its language.

Careful:

Q. Can *every* language be recognised* by an NFA?
(*Recall that "recognising" is a synonym for "accepting".)

On the Power of Non-Determinism!

Q. Is there a language that is accepted by an NFA for which we *cannot* find a DFA that (also) accepts it?

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A. No.

Theorem.
If language L is accepted by a NFA, then there is some DFA which accepts the same language. Or more formally:
Let A be an NFA. Then, there exists a DFA A', such that $L(A) = L(A')$.

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Proof.
We provide an algorithm that, given an arbitrary NFA A, creates a DFA A', such that $L(A) = L(A')$. (In the worst-case, it might take exponential time.)

Determinisation of NFAs

NFA to DFA Construction

Assumption. We have an NFA with state set $\{q_0, \dots, q_n\}$.

Basic Idea.

- consider all possible runs of the NFA in parallel
- as a consequence, can be in a *set* of states

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- A *state* of the DFA is a *set of states* of the NFA:
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- DFA *final states* are state sets that *contain* a final NFA state.

Subset Construction: The Finer Points

Input. Let NFA $A = (\Sigma, S, s_0, F, R)$.

Subset Construction.

- DFA states are *subsets* of S but each subset plays the role of a single state!

Subset Construction: The Finer Points

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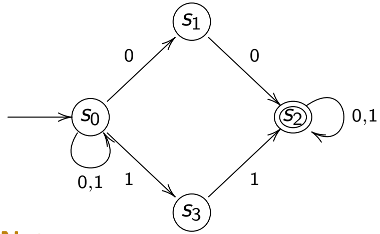
$\rightarrow$ Then, we get $\{q_3, q_7\} \xrightarrow{0} \{q_3, q_5, q_8\}$ and $\{q_3, q_7\} \xrightarrow{1} \{q_{42}\}$

Determinisation: Example

The "double digits" automaton



Subset Construction: transition table

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$\to \{s_0\}$		

Note.

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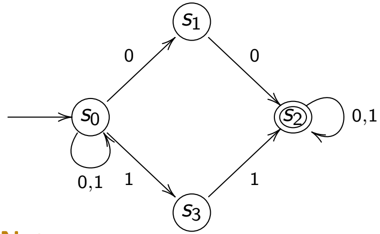

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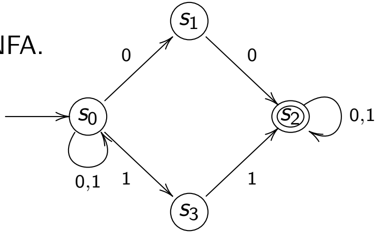
- Once the table is complete replace each DFA state set by a simple name

Determinisation Example, as Diagrams

Double Digits, as NFA.



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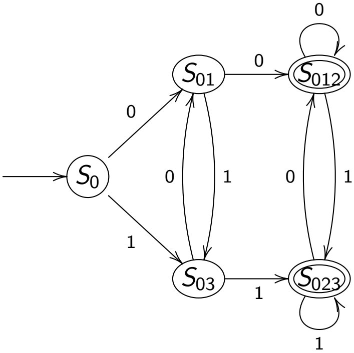

NFAs with ε-transitions

More Expressive Power: ϵ-transitions

Extra Ingredient: Spontaneous transitions that don't "consume" a letter

- NFAs that may change state *without* consuming a symbol.
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- can convert NFAs with ϵ-transitions to (standard) NFAs

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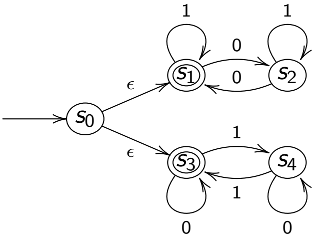
Formal Definition. An NFA with ϵ-transitions is an NFA, but the transition relation has the form

$$R \subseteq S \times \Sigma \cup \{\epsilon\} \times S$$

- cf. NFAs with transition relation $R \subseteq S \times \Sigma \times S$
- $R(s, \epsilon, s')$ is a spontaneous transition (without reading input symbol)
- ϵ is *not* an element of the alphabet!

ϵ-NFA: Example

General Pattern. ϵ-transitions say "or"



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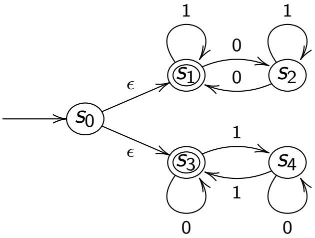


Q. What does that automaton do?

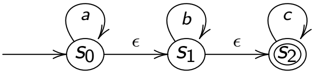
A. Interpretation:

- "top" automaton (with start state s_1) requires even number of 0's
- "bottom" automaton (with start state s_3) requires even number of 1's

$\rightarrow$ entire automaton (with start state s_0) accepts *either* an even number of 1's *or* an even number of 0's

Example and Acceptance

Language of this Automaton?



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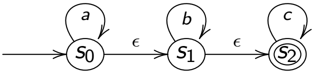

Acceptance Informally. An ϵ-NFA A *accepts* a word $w = x_1 \dots x_n$ if there is a sequence of states

$$s_0 \xrightarrow{\epsilon^*} s_1 \xrightarrow{x_1} s_1' \xrightarrow{\epsilon^*} s_2 \xrightarrow{x_2} s_2' \dots s_n \xrightarrow{x_n} s_n' \xrightarrow{\epsilon^*} f$$

where s_0 is the starting state, $f \in F$ is an accepting state and

- $s_i \xrightarrow{x} s_j$ if there is an x-transition from s_i to s_j, i.e., $(s_i, x, s_j) \in R$
- $s_i \xrightarrow{\epsilon^*} s_j$ if there is a sequence of ϵ-transitions from s_i to s_j.

In particular: the empty string $\epsilon \in L(A)$ if $s_0 \xrightarrow{\epsilon^*} f$ for a final state $f \in F$.

Eventual State Relation for ϵ-NFAs

ϵ-closure. For an ϵ-NFA (Σ, S, s_0, F, R), the *ϵ-closure* of a state $s \in S$ is given by: $\quad \text{eclose}(s) = \{s' \in S \mid \text{ there is a sequence of } \epsilon\text{-transitions from } s \text{ to } s'\}$
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Acceptance (and language) for DFAs / NFAs:
A string w is *accepted* by an ϵ-NFA A (in symbols: $w \in L(A)$) if $(s_0, w, f) \in R^*$ for some final state $f \in F$, that is

$$L(A) = \{w \in \Sigma^* \mid \exists f \in F.(s_0, w, f) \in R^*\}$$

Relationship Between NFAs and ϵ-NFAs

Q. Are there languages *only* accepted by ϵ-NFAs?

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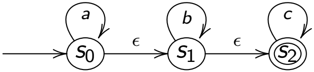
- Put an arc $s \xrightarrow{x} t$ *into A'* if there is some $s' \in \text{eclose}(s)$, such that $s' \xrightarrow{x} t$ *in A*. Formally:

$$R' = \{(s, x, t) \mid (s', x, t) \in R \text{ for some } s' \in \text{eclose}(s)\}$$

(double-check that A and A' accept the same strings!)

Example for ϵ-Elimination



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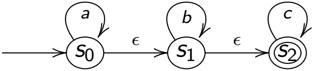

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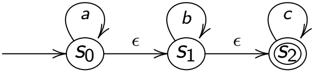

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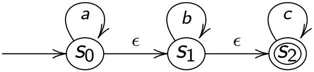

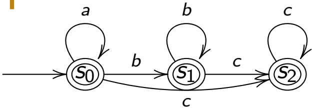

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- The language of bit-strings that have an even number of 1s: $0^*(10^*10^*)^*$
  (Zero is even, so $0 \dots 0$ should be accepted. Thus, $(0^*10^*10^*)^*$ is wrong)

Regular Expressions — More Examples

- A single zero or binary numerals without leading zero: $0|(1(0|1)^*)$
- The set of strings over $\{a, b, c\}$ with just one c: $(a \mid b)^* c (a \mid b)^*$
- The language of bit-strings that have an even number of 1s: $0^*(10^*10^*)^*$
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  The binary fractional numerals between 0 and 1 with no trailing zeroes. E.g. 1, 0, 0.1, 0.110011 but not .1 or 0.10. The last 1 in the expression is required to prevent redundant zeros at the end.

The Definition of Regular Expressions

Key Concept.

- regular expressions are purely *syntactical* – just like formulae
- *but:* every expression denotes a set of strings – this is the meaning.

Definition.

The regular expressions over alphabet Σ and the sets that they denote are:

- $\emptyset$ is a regular expression and denotes the empty set $\emptyset$
- ϵ is a regular expression and denotes the set $\{\epsilon\}$
- for each $a \in \Sigma$, a is a regular expression and denotes the set $\{a\}$

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- for each $a \in \Sigma$, a is a regular expression and denotes the set $\{a\}$
- If α and β are regular expressions denoting languages R and S resp., then:
  - $\alpha \mid \beta$ denotes $R \cup S$
  - $\alpha\,\beta$ denotes RS which is $\{ww' \mid w \in R \land w' \in S\}$
  - α^* denotes R^*, i.e., the set of *finitely* many $r_i \in R$, concatenated, i.e., R^* is (inductively) defined as $\{\epsilon\} \cup RR^*$

Regular Expressions and DFAs

Key Insights.

- For every DFA A, we have regular expression r with $L(A) = L(r)$.
  (We didn't show or even state that yet!)
- For every regular expression r, we have a DFA A with $L(r) = L(A)$.
  (You will see this in the next few slides.)
- Recall that we already showed the equivalence of DFAs and NFAs.
- Thus, the "power" of NFAs / DFAs are *completely described* by regular expressions (and vice versa). In other words:

  DFAs, NFAs, and regular expressions are all equally expressive.

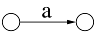
Regular Expressions to ϵ-NFAs

Key Insight.

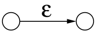
- regular expressions are an *inductively defined structure*
- e.g., representable by an inductive data type in Haskell
- as a consequence, we can give *inductive definition* of the corresponding automaton

Construction. (start state on left, final state on right)

- When the regular expression is a symbol a of the alphabet (language is $\{a\}$) the automaton is



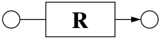
- When the regular expression is ϵ (language is $\{\epsilon\}$) the automaton is



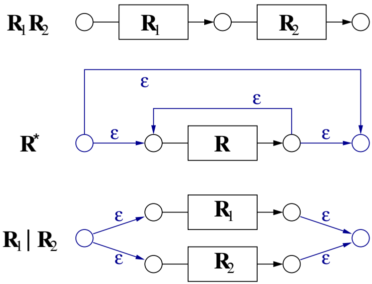
- When the regular expression is $\emptyset$ (language is $\emptyset$) the automaton has no edges

Regular Expressions to NFAs, ctd

Suppose the NFA corresponding to some regular expression R is:



Then, we can inductively define the NFAs corresponding to composite regular expressions as follows:



Example

Given the regular expression for binary numerals without leading zeros, $(0 \mid 1(0|1)^*)$, the previous algorithm (the inductive definition) gives this NFA:

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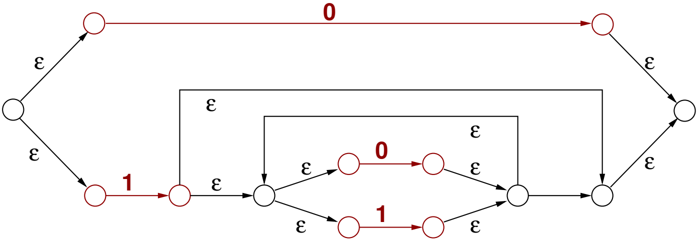

Closing the Loop

Given a finite alphabet Σ and a language $L \subseteq \Sigma^*$. The following are equivalent:

- L can be described by a regular expression
- L can be recognised by an ϵ-NFA
- L can be recognised by an NFA
- L can be recognised by a DFA ...

as we can convert regular expressions into ϵ-NFAs into NFAs into DFAs.

Missing Link.

Construction of regular expressions from DFAs (not covered in this course).

Summary

Summary.

Starting Point. Finite Automata

- motivated by computers having finite memory (only)
- solving simple problems: is string s accepted?

Limitations of Finite Automata

- Some languages can't be recognised, e.g., $L = \{a^n b^n \mid n \geq 0\}$

Characterisation of expressive power

- can go back and forth between automata and regular expressions

Q. Are finite automata a "good" model of computation?

- if yes, why?
- if not, why not? What is missing?

Literature.

- *Introduction to Automata Theory, Languages, and Computation*
  By Hopcroft, Motwani, and Ullman.

  A classic text that has been re-worked from a standard textbook.
- *Introduction To The Theory Of Computation*
  by Michael Sipser

  The part on Automata and Languages covers (more than) what we have discussed here.
- There are tons of exercises one can practice with.
  (Online and in our repository.)