COMP1600, week 9: Grammars and Push-down Automata (PDAs)

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slides based on those by: Dirk Pattinson
(with contributions by Victor Rivera and previous colleagues)

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Australian
National
University

Overview of Week 9

- Introduction
- Grammars and Derivations
- The Chomsky Hierarchy
- Regular Grammars/Languages
- Context-Free Grammars/Languages
- Parse-trees and Ambiguity
- Pushdown Automata

Introduction

Terminology Recap: Formal Languages

- The **alphabet** or **vocabulary** of a formal language is a set of **tokens** (or **letters**). It is usually denoted Σ.
- A **string** over Σ is a *sequence* of tokens.
    - e.g., sequence may be empty, giving empty string ϵ
    - e.g., $ababc$ is a string over $\Sigma = \{a, b, c\}$

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  - e.g., sequence may be empty, giving empty string ϵ
  - e.g., *ababc* is a string over $\Sigma = \{a, b, c\}$
- A **language** with alphabet Σ is some set of strings over Σ.
  - e.g., the set of all strings Σ^*
  - e.g., the set of all strings of even length, $\{w \in \Sigma^* \mid |w| \ \% \ 2 \ == 0\}$
  - e.g., ... we had many examples in the last two weeks!
  - e.g., *any* subset of Σ^*

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Grammar.

- a concept that has been invented in linguistics to describe natural languages
- describes how strings are *constructed* rather than how membership can be *checked* (e.g., by an automaton; though does it? what does a RegEx do?)
- *the* main tool to describe syntax; grammars are extremely important!

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- *Fun fact:* My expertise on Hierarchical Planning (a subfield of Artificial Intelligence) largely overlaps with formal grammars!

Grammars and Derivations

Formal Grammars

Formal Definition. A *grammar* is a quadruple $\langle V_t, V_n, S, P\rangle$ where

- V_t is a finite set of *terminal symbols* (the alphabet)
- V_n is a finite set of **non-terminal symbols** disjoint from V_t (Notation: $V = V_t \cup V_n$)
- S is a distinguished non-terminal symbol called the *start symbol*
- P is a set of *productions* (also called *production rules*), each written

$$\alpha \to \beta$$

  where
  - $\alpha \in V^* V_n V^*$ (i.e., at least one non-terminal in α)
  - $\beta \in V^*$ (i.e., β is *any* sequence of symbols)

Example (for Syntax)

The grammar (recall: grammars have the form $\langle V_t, V_n, S, P \rangle$)

$$G = \langle \{a, b\},\ \{S, A\},\ S,\ \{S \rightarrow aAb,\ aA \rightarrow aaAb,\ A \rightarrow \epsilon\} \rangle$$

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$$A \to \epsilon$$

Notation.

- Often, we just list the productions P, as all other components can be inferred (S is the standard notation for the start symbol)
- The notation $\alpha \to \beta_1 \mid \cdots \mid \beta_n$ abbreviates the *set* of productions

$$\alpha \to \beta_1, \quad \alpha \to \beta_2, \quad \ldots, \quad \alpha \to \beta_n$$

(like for inductive data types)

Derivations

Intuition.

- A production $\alpha \rightarrow \beta$ tells you what you can "make" if you have α: you can turn it into β. It allows us to re-write any string $\gamma\alpha\rho$ to $\gamma\beta\rho$.
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- formally: $S(G) = \{w \in V^* \mid S \Rightarrow^* w\}$.

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Language of a grammar.

- informally: all strings of terminal symbols that can be generated from the start symbol S
- formally: $L(G) = \{w \in V_t^* \mid S \Rightarrow^* w\}$, which is a subset of $S(G)$
- that's the same as $L(G) = S(G) \cap V_t^*$

Example

Productions of the grammar G.

$$S \to aAb, \qquad aA \to aaAb, \qquad A \to \epsilon.$$

Example Derivation.

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$$S \Rightarrow aAb \Rightarrow aaAbb \Rightarrow aaaAbbb \Rightarrow aaabbb$$

- last string $aaabbb$ is a *word*, others are *sentential forms*

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$$L(G) = \{a^n b^n \mid n \in \mathbb{N},\ n \geq 1\}$$

Alternative Grammar for the *same* language

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The Chomsky Hierarchy

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Each grammar is of a *type*: (There are *lots* of intermediate types, too.)

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This also gives us a way to classify *languages*. (Next slide.)

Classification of Languages

Definition. A language is *type n* if it can be generated by a type n grammar.

Immediate Fact.

- Every language of type $n + 1$ is also of type n.
- E.g., every context-free language (type 2) is also context-sensitive (type 1).

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Disproving that a language is of type n

- must show that *no* type n-grammar generates the language
- usually a *difficult* problem. (There are complex theorems to help.)

Example – Language $\{a^n b^n \mid n \in \mathbb{N}, n \geq 1\}$

Different grammars for this language

- *Unrestricted (type 0):*

$$\begin{aligned} S &\to aAb \\ aA &\to aaAb \\ A &\to \epsilon \end{aligned}$$

- *Context-free (type 2):*

$$\begin{aligned} S &\to ab \\ S &\to aSb \end{aligned}$$

Recall. We know from last week that there is no DFA accepting L

- We will see that this means that there's no regular grammar
- so the language is context-free, but not regular.

Regular (Type 3) Grammars

Definition. A grammar is *regular* if all its productions are either *right-linear*, i.e. of the form

$$A \to aB \qquad \text{or} \qquad A \to a \qquad \text{or} \qquad A \to \epsilon$$

or *left-linear*, i.e., of the form

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- right and left linear grammars are equivalent: they generate the same languages (and can hence be turned into each other)
- we focus on *right linear* ones (it's probably slightly more intuitive)
- i.e., one symbol is *generated* at a time (cf. DFA/NFA!)
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Next Goal. Relate regular languages to DFAs/NFAs/RegExs.

Regular Grammars/Languages

Regular Languages – Many Views

Theorem. Let L be a language. Then the following are equivalent:

- L is the language generated by a *right-linear grammar*;
- L is the language generated by a *left-linear grammar*;
- L is the language accepted by some *DFA*;
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So far.

- have seen that NFAs and DFAs generate the same languages
- have shown that regular expressions can be turned into NFAs (hence DFAs)
- claimed that DFAs can be turned into regular expressions

Goal.

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Goal. Show that NFAs and right-linear grammars generate the same languages.

From NFAs to Right-linear Grammars

Given. Take an NFA $A = (\Sigma, S, s_0, F, R)$.

- alphabet, state set, initial state, final states, transition relation

Construction of a right-linear grammar

- *terminal symbols* are elements of the alphabet Σ;
- *non-terminal symbols* are the states S;
- *start symbol* is the start state s_0;
- *productions* are constructed as follows:

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Each *transition*	gives *production*
$T \xrightarrow{a} U$	$T \rightarrow aU$
Each *final state*	gives *production*
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(Formally, a transition $T \xrightarrow{a} U$ means $(T, a, U) \in R$.)

Observation.

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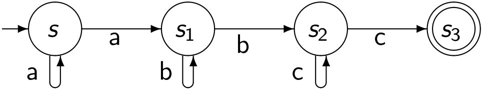
Observation. The grammar so generated is right-linear, and hence regular.

NFAs to Right-linear Grammars – Example

Given. A non-deterministic automaton



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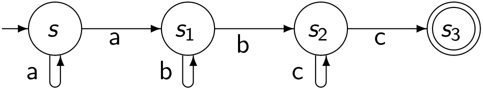

Equivalent Grammar obtained by construction

$$S \to aS \qquad S_2 \to cS_2$$
$$S \to aS_1 \qquad S_2 \to cS_3$$
$$S_1 \to bS_1 \qquad S_3 \to \varepsilon$$
$$S_1 \to bS_2$$

(We capitalised the NFA states to make clear they are non-terminals!)

Exercise. Convince yourself that the NFA accepts precisely the words that the grammar generates. (We still owe a proof of correctness of the translation.)

From Right-linear Grammars to NFAs

Given. Right-linear grammar (V_t, V_n, S, P)

- terminals, non-terminals, start symbol, productions

Construction of an equivalent NFA has

- *alphabet* is the terminal symbols V_t;
- *states* are the *non-terminal symbols* V_n plus new state S_f (for final);
- *start state* is the start symbol S;
- *final states* are S_f and *all* non-terminals T such that there exists a production $T \rightarrow \varepsilon$;
- *transition relation* is constructed as follows:

Each *production*	gives *transition*
$T \rightarrow aU$	$T \xrightarrow{a} U$
$T \rightarrow a$	$T \xrightarrow{a} S_f$

Right-linear Grammars to NFAs – Ex.

Given. Grammar G with the productions

$$S \to 0 \qquad S \to 1T$$

$$T \to \varepsilon \qquad T \to 0T \qquad T \to 1T$$

(generates the language

Right-linear Grammars to NFAs – Ex.

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(generates the language of binary strings without leading zeros)

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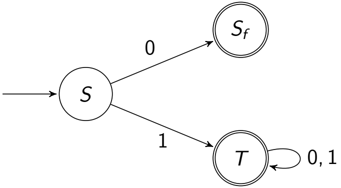

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Context-Free Grammars/Languages

Context-Free (Type 2) Grammars (CFGs)

Recall. A grammar is type-2 or *context-free* if all productions have the form

$$A \to w$$

where $A \in V_n$ is a non-terminal, and $w \in V^*$ is an (arbitrary) string.

- left side is non-terminal
- right side can be anything
- *independent* of context, replace LHS (left hand side) with RHS (right HS).

In Contrast. Context-Sensitive grammars may have productions

$$\alpha A \beta \to \alpha w \beta$$

which may only replace A by w if A appears in context α_β

Example

Goal. Design a CFG for the language

$$L = \{a^m b^n c^{m-n} \mid m \geq n \geq 0\}$$

Strategy. First, understand the language!

Example

Goal. Design a CFG for the language

$$L = \{a^m b^n c^{m-n} \mid m \geq n \geq 0\}$$

Strategy. First, understand the language! Every word $w \in L$ can be split

$$w = a^{m-n} a^n b^n c^{m-n}$$

and hence $L = \{a^k a^n b^n c^k \mid n, k \geq 0\}$

Resulting Grammar.

Example

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and hence $L = \{a^k a^n b^n c^k \mid n, k \geq 0\}$

- convenient to *not* have comparison between n and m
- generate $a^k \dots c^k$, i.e., same number of leading as and trailing c^s
- fill $\dots$ in the middle by $a^n b^n$, i.e., same number of as and bs
- use different non-terminals for both phases of the construction

Resulting Grammar. (productions only)

$$\begin{aligned} S &\rightarrow aSc \mid T \\ T &\rightarrow aTb \mid \epsilon \end{aligned}$$

Example cont'd

Grammar

$$\begin{aligned} S &\rightarrow aSc \mid T \\ T &\rightarrow aTb \mid \epsilon \end{aligned}$$

Example Derivation. of $aaabbc$:

$$\begin{aligned} S &\Rightarrow aSc \\ &\Rightarrow aTc \\ &\Rightarrow aaTbc \\ &\Rightarrow aaaTbbc \\ &\Rightarrow aaabbc \end{aligned}$$

The Power of Context-Free Grammars

A fun example (for the “usefulness” of CFGs):
`http://pdos.csail.mit.edu/scigen`

This tool generates (fake) scientific papers based on formal grammars!

Note that:

- Sadly, the generator doesn’t seem to work anymore.
- But the page mentions where their (fake papers) were accepted!
- They link similar “services”.
- Maybe you can still find a working version somewhere...

Parse Trees and Ambiguity

Parse Trees

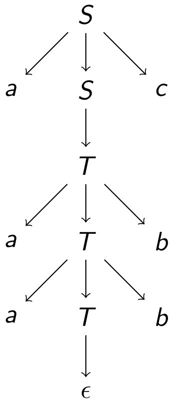
Idea. Represent derivation as *tree* rather than as list of rule applications

- describes where and how productions have been applied
- generated word can be collected at the leaves

Example for the grammar that we have just constructed

$$S \rightarrow aSc \mid T$$
$$T \rightarrow aTb \mid \epsilon$$

word: $w = aaabbc$



Parse Trees Carry Semantics

Take the code

```
if e1 then if e2 then s1 else s2
```

where **e1**, **e2** are boolean expressions and `s1`, `s2` are subprograms.

Two Readings.

```
if e1 then ( if e2 then s1 else s2 )
```

and

```
if e1 then ( if e2 then s1 ) else s2
```

Goal. *unambiguous* interpretation of the code leading to *determined* and *clear* program execution.

Ambiguity

Recall that we can present CFG derivations as *parse trees*.

Until now this was merely a pretty presentation; now it will become important.

Definitions:

- A context-free grammar G is **unambiguous** iff every string can be derived by ***at most*** one parse tree.
- G is **ambiguous** iff there exists a word $w \in L(G)$ derivable by more than one parse tree.

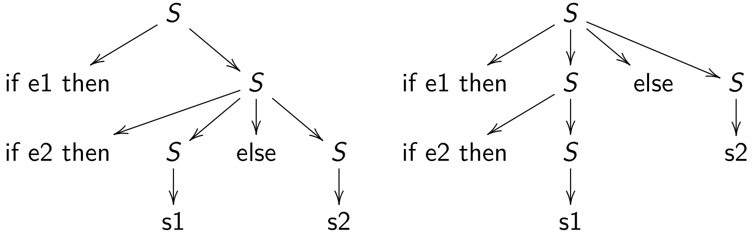
Example: If-Then and If-Then-Else

Consider the CFG

$$S \quad \rightarrow \quad \text{if } \textbf{bexp} \text{ then } S \mid \text{if } \textbf{bexp} \text{ then } S \text{ else } S \mid \textbf{prog}$$

where **bexp** and **prog** stand for boolean expressions and if statement-free programs respectively, defined elsewhere.

The string if e1 then if e2 then s1 else s2 then has two parse trees:



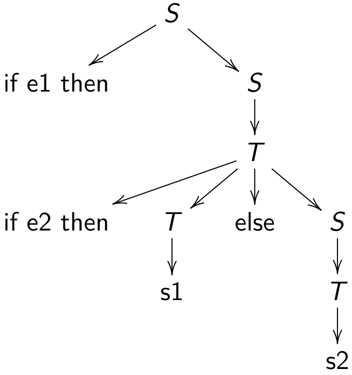
Example: If-Then and If-Then-Else

That grammar was **ambiguous**. But here's a grammar accepting the *exact same language* that is **unambiguous**:

$$S \rightarrow \text{ if } \textbf{bexp} \text{ then } S \mid T$$
$$T \rightarrow \text{ if } \textbf{bexp} \text{ then } T \text{ else } S \mid \textbf{prog}$$

There is now **only one** parse tree for if e1 then if e2 then s1 else s2.
(Given on the next slide.)

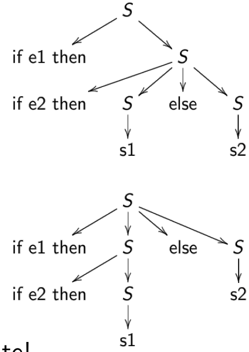
Example: If-Then and If-Then-Else



You **cannot** parse this string as if “e1 then (if e2 then s1) else s2”.

Q. Does that mean we can’t generate the ‘meaning’ of:
“e1 then (if e2 then s1) else s2”?

Reflecting on This Example



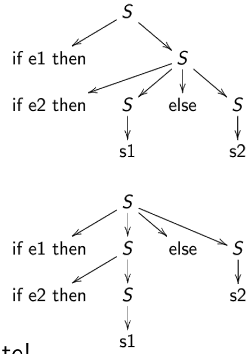
Observation.

- there's more than one grammar for a language
- some are ambiguous, others are not
- ambiguity is a property of *grammars*

Grammars for Programs.

- ambiguity is bad: don't know how program will execute!

Reflecting on This Example



Observation.

- there's more than one grammar for a language
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- ambiguity is a property of *grammars*

Grammars for Programs.

- ambiguity is bad: don't know how program will execute!
- replace ambiguous grammar with unambiguous one

Choices for converting ambiguous grammars to unambiguous ones

- *decide* on just *one* parse tree
- e.g. `if e1 then ( if e2 then s1 ) else s2` vs.
  `if e1 then ( if e2 then s1 else s2 )`
- in example: we have *chosen*:
  `if e1 then ( if e2 then s1 ) else s2`

What Ambiguity Isn't

Q. Is the grammar with the following production ambiguous?

$$T \quad \rightarrow \quad \text{if } \textbf{bexp} \text{ then } T \text{ else } S$$

Reasoning.

- Suppose that the above production was used
- we can then expand either T or S first.

What Ambiguity Isn't

Q. Is the grammar with the following production ambiguous?

$$T \quad \rightarrow \quad \text{if } \mathbf{bexp} \text{ then } T \text{ else } S$$

Reasoning.

- Suppose that the above production was used
- we can then expand either T or S first.

A. This is *not* ambiguity.

- both options give rise to the *same* parse tree
- indeed, for context-free languages it *doesn't* matter what production is applied first.
- thinking about parse trees, both expansions happen in parallel.

Main Message. Parse trees provide a better representation of syntax than derivations.

Inherently Ambiguous Languages

Q1. Can we always remove ambiguity?

Example. Language $L = \{a^i b^j c^k \mid (j = i \text{ or } j = k) \text{ and } i, j, k \in \mathbb{N}\}$

Q2. Why is this context-free?

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Q2. Why is this context-free?

A. Note that $L = \{a^i b^i c^k \mid i, k \in \mathbb{N}\} \cup \{a^i b^j c^j \mid i, j \in \mathbb{N}\}$

- idea: start with production that "splits" between the union
- $S \to T \mid W$ where T is "left" and W is "right"

Complete Grammar. It starts with $S \ \to \ T \ \mid \ W$. Assume:

- left part uses non-terminals T, U, V
- right part uses non-terminals W, X, Y

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$$\begin{array}{lll} T & \to & UV \\ U & \to & aUb \mid \epsilon \\ V & \to & cV \mid \epsilon \end{array} \qquad \begin{array}{lll} W & \to & XY \\ X & \to & aX \mid \epsilon \\ Y & \to & bYc \mid \epsilon \end{array}$$

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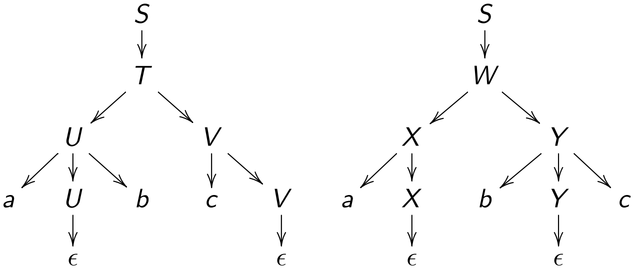
$$\begin{aligned} T &\to UV & \qquad W &\to XY \\ U &\to aUb \mid \epsilon & \qquad X &\to aX \mid \epsilon \\ V &\to cV \mid \epsilon & \qquad Y &\to bYc \mid \epsilon \end{aligned}$$

Q3. Why is this *language* ambiguous?

Inherently Ambiguous Languages

Problem. Both left part $a^i b^i c^k$ and right part $a^i b^j c^j$ has non-empty intersection: $a^i b^i c^i$

Ambiguity where a, b and c are equi-numerous, e.g., $a^1 b^1 c^1 = abc$:



So there are two parse trees for the same word!

Fact. There is *no* unambiguous grammar for this language (we don't prove this)

The Bad News

Q1. Can we *compute* an unambiguous grammar whenever one exists?

Q2. Can we even *determine* whether an unambiguous grammar exists?

A. If we interpret "compute" and "determine" as "by means of a program" (that works for an arbitrary CFL), then no.

- There is *no* program that solves this problem for *all* grammars
- input: CFG G, output: ambiguous or not. This problem is *undecidable*

(More undecidable problems next week!)

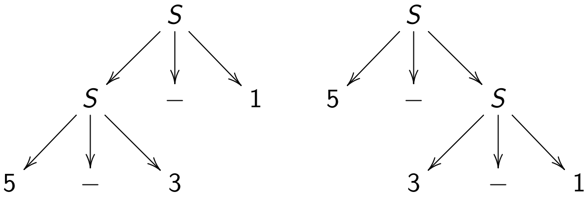
Example: Subtraction

Example.

$$S \quad \to \quad S - S \mid \mathrm{int}$$

- int stands for integers
- the intended meaning of $-$ is subtraction

Ambiguity.



Evaluation.

- left parse tree evaluates to 1
- right parse tree evaluates to 3
- so ambiguity matters! (As we also saw for the if/else statements.)

Technique 1: Associativity

Idea for ambiguity induced by binary operator (think: $-$)

- prescribe "implicit parentheses", e.g. $a - b - c \equiv (a - b) - c$
- make operator associate to the left or the right

Left Associativity.

$$S \quad \rightarrow \quad S - \textbf{int} \mid \textbf{int}$$

Result.

- $5 - 3 - 1$ can only be read as $(5 - 3) - 1$
- this is *left associativity*

Right Associativity.

$$S \rightarrow \textbf{int} - S \mid \textbf{int}$$

Idea. Break the symmetry

- one side of operator forced to lower level
- here: force right hand side of i to lower level
- create example derivation trees for all three grammars to see why that helps

Example: Multiplication and Addition

Example. Grammar for addition and multiplication

$$S \quad \to \quad S * S \mid S + S \mid \textbf{int}$$

Ambiguity.

- $1 + 2 * 3$ can be read as $(1 + 2) * 3$ and $1 + (2 * 3)$ with different results
- also $1 + 2 + 3$ is ambiguous – but this doesn't matter here.

Take 1. The trick we have just seen

- strictly evaluate from left to right

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Take 1. The trick we have just seen

- strictly evaluate from left to right
- but this gives $1 + 2 * 3 \equiv (1 + 2) * 3$, *not* intended!

Goal. Want $*$ to have *higher precedence* than $+$

Technique 2: Precedence

Example Grammar giving $*$ higher precedence:

$$\begin{aligned} S &\to \; S + T \;\mid\; T \\ T &\to \; T * \textbf{int} \;\mid\; \textbf{int} \end{aligned}$$

Given e.g. $1 + 2 * 3$ or $2 * 3 + 1$

- *forced* to expand $+$ first: otherwise only $*$
- so $+$ will be *last* operation evaluated

Example. Derivation of $1 + 2 * 3$ (which we want to interpret as $1 + (2 * 3)$)

- suppose we start with $S \Rightarrow T \Rightarrow T * \mathrm{int}$
- stuck, as cannot generate $1 + 2$ from T

Idea. Forcing operation with *higher* priority to *lower* level

- three levels: S, (highest), T (middle) and integers
- lowest-priority operation generated by highest-level non-terminal

Example: Basic Arithmetic

Repeated use of $+$ and $*$:

$$\begin{aligned} S &\rightarrow S + T \mid S - T \mid T \\ T &\rightarrow T * U \mid T/U \mid U \\ U &\rightarrow (S) \mid \textbf{int} \end{aligned}$$

Main Differences.

- have *parentheses* to break operator priorities, e.g. $(1 + 2) * 3$
- parentheses at *lowest* level, so *highest* priority
- lower-priority operator can be inside parentheses
- expressions of arbitrary complexity (no nesting in previous examples)

Example: Balanced Brackets

$$S \quad \rightarrow \quad \epsilon \mid (S) \mid SS$$

Ambiguity.

- associativity: create brackets from left or from right (as before).
- at least wo ways of generating ():
  - $S \Rightarrow SS \Rightarrow S \Rightarrow (S) \Rightarrow ()$ and
  - $S \Rightarrow (S) \Rightarrow ()$
- indeed, *any* expression has *infinitely many* parse trees

Reason. More than one way to derive ϵ.

Technique 3: Controlling ϵ

Alternative Grammar with only *one* way to derive ϵ:

$$\begin{aligned} S &\to \epsilon \mid T \\ T &\to TU \mid U \\ U &\to () \mid (T) \end{aligned}$$

- ϵ can only be derived from S
- all other derivations go through T
- here: combined with multiple level technique
- ambiguity with ϵ can be hard to miss!

Pushdown Automata

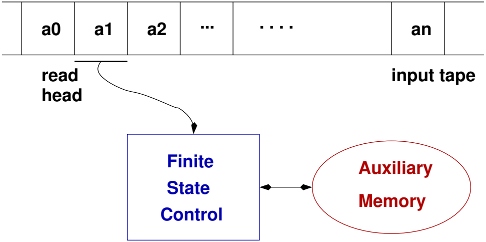
From Grammars to Automata

So Far.

- regular languages correspond to regular grammars (by definition).
- regular languages are exactly those accepted by FSAs or regular expressions (or regular grammars, of course).

Q. What automata correspond to *context-free* grammars?

General Structure of Automata



- *input tape* is a set of symbols
- *finite state control* is just like for DFAs / NFAs
- symbols are processed and head advances
- new aspect: *auxiliary memory*

Auxiliary Memory classifies languages and grammars

- no auxiliary memory: NFAs / DFAs: regular languages
- *stack*: *push-down automata*: context-free languages
- *unbounded tape*: Turing machines: all languages

PDAs cont'd

Actions of a push-down automaton

- change of internal state
- pushing or popping the stack
- advance to next input symbol

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Action dependencies. Actions generally depend on

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Acceptance. The machine accepts if

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PDAs cont'd

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Variation.

- PDAs can *equivalently* be defined without final states F.
- Then, acceptance condition is having an empty stack (after the input word was completely read). **But we don't use that!**

Example

Language (that cannot be recognised by a DFA)

$$L = \{a^n b^n \mid n \geq 1\}$$

- *cannot* be recognised by a DFA
- *can* be generated by a context-free grammar
- *can* be recognised by a PDA

PDA design. (ad hoc, but showcases the idea)

- *phase 1:* (state s_1) *push* a's from the input onto the stack
- *phase 2:* (state s_2) *pop* a's from the stack, if there is a b on input
- *finalise:* if the input is exhausted and the stack is empty, enter a final state (s_3), i.e., accept the string.

Deterministic PDA – Definition

Definition. A *deterministic PDA* has the form $(S, s_0, F, \Sigma, \Gamma, Z, \delta)$, where

- S is the finite set of *states*, $s_0 \in S$ is the *initial state* and $F \subseteq S$ are the *final states*;
- Σ is the finite *alphabet*, or set of *input symbols*;
- Γ is the finite set of *stack symbols*, and $Z \in \Gamma$ is the *initial stack symbol*;
- δ is a (partial) *transition function*

$$\begin{array}{rcl} \delta \ : \ S \times (\Sigma \cup \{\epsilon\}) \times \Gamma & \nrightarrow & S \times \Gamma^* \\ \delta \ : \ (\text{state}, \text{input token or } \epsilon, \text{top-of-stack}) & \nrightarrow & (\text{new state}, \text{new top of stack}) \end{array}$$

Additional Requirement to ensure determinism:

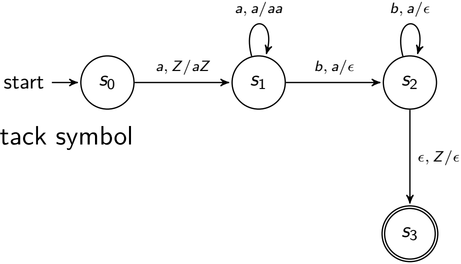
- if $\delta(s, \epsilon, \gamma)$ is defined, then $\delta(s, x, \gamma)$ is undefined for all $x \in \Sigma$ and $\gamma \in \Gamma$
- ensures that automaton has *at most* one execution

Notation

Given. Deterministic PDA with transition function

$$\delta \ : \ S \times (\Sigma \cup \{\epsilon\}) \times \Gamma \quad \nrightarrow \quad S \times \Gamma^*$$

$$\delta \ : \ (\text{state}, \text{input token or } \epsilon, \text{top-of-stack}) \quad \nrightarrow \quad (\text{new state}, \text{new top of stack})$$



Notation.

- write $\delta(s, x, \gamma) = s'/\sigma$
- $\sigma \in \Gamma^*$ is a *string* that replaces top stack symbol
- *final states* are usually underlined ($\underline{s}$)

Rationale.

- *replacing* top stack symbol gives just *one* operation for push and pop
- pop: $\delta(s, x, \gamma) = s'/\epsilon$
- push: $\delta(s, x, \gamma) = s'/w\gamma$

Two types of PDA transition

Input-consuming transitions

- δ contains $(s_1, x, \gamma) \mapsto s_2/\sigma$
- automaton reads symbol x
- symbol x is consumed

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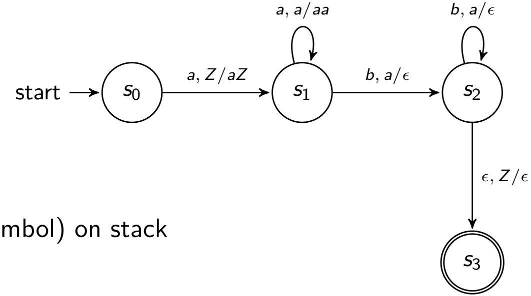
- automaton reads symbol x
- symbol x is consumed

Non-consuming transitions

- independent of input symbol
- can happen *any time* and does not consume input symbol
- δ contains $(s_1, \epsilon, \gamma) \mapsto s_2/\sigma$
  Recall that for the pair s_1, γ, we can't have any other entry (s_1, x, γ) with $x \in \Sigma$ to stay deterministic! (See slide 50)
  Q. How is this different from epsilon transitions in ϵ-NFAs?

Example cont'd



Language $L = \{a^n b^n \mid n \geq 1\}$

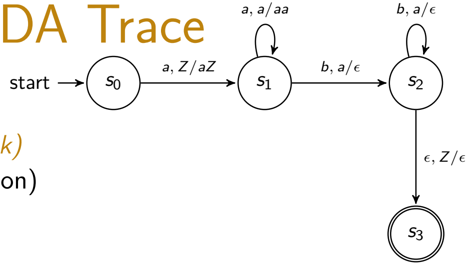
Push-down automaton

- starts with Z (initial stack symbol) on stack
- final state is s_3 (underlined)
- transition function (partial) given by

$$\begin{array}{lcll}
\delta(s_0, a, Z) & \mapsto & s_1/aZ & \text{push first } a \\
\delta(s_1, a, a) & \mapsto & s_1/aa & \text{push further } a\text{'s} \\
\delta(s_1, b, a) & \mapsto & s_2/\epsilon & \text{start popping } a\text{'s} \\
\delta(s_2, b, a) & \mapsto & s_2/\epsilon & \text{pop further } a\text{'s} \\
\delta(s_2, \epsilon, Z) & \mapsto & \underline{s_3}/\epsilon & \text{accept}
\end{array}$$

(δ is partial, i.e., undefined for many arguments)
Also note that we don't have to delete Z in the last step.
The stack doesn't have to be empty at the end.

Example cont'd — PDA Trace



PDA configurations

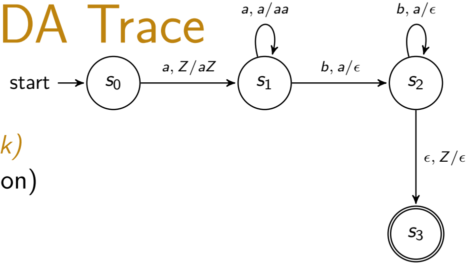
- triples: *(state, remaining input, stack)*
- top of stack on the *left* (by convention)

Example Execution.

$$(s_0, aaabbb, Z) \Rightarrow$$

(accept)

Example cont'd — PDA Trace



PDA configurations

- triples: *(state, remaining input, stack)*
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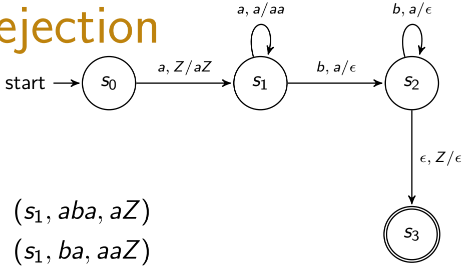
Example Execution.

$$
\begin{aligned}
(s_0, aaabbb, Z) &\Rightarrow (s_1, aabbb, aZ) && \text{(push first } a\text{)}\\
&\Rightarrow (s_1, abbb, aaZ) && \text{(push further } a\text{'s)}\\
&\Rightarrow (s_1, bbb, aaaZ) && \text{(push further } a\text{'s)}\\
&\Rightarrow (s_2, bb, aaZ) && \text{(start popping } a\text{'s)}\\
&\Rightarrow (s_2, b, aZ) && \text{(pop further } a\text{'s)}\\
&\Rightarrow (s_2, \epsilon, Z) && \text{(pop further } a\text{'s)}\\
&\Rightarrow (\underline{s_3}, \epsilon, \epsilon) && \text{(accept)}
\end{aligned}
$$

Accepting execution. Input exhausted, ends in final state (as usual!).

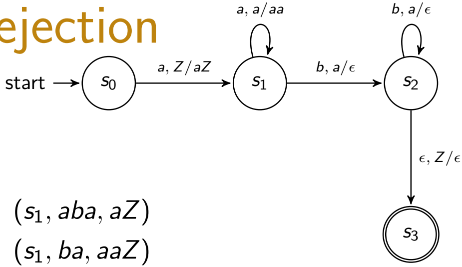
Example cont'd — Rejection



PDA execution.

$$\begin{aligned}(s_0, aaba, Z) &\Rightarrow (s_1, aba, aZ)\\ &\Rightarrow (s_1, ba, aaZ)\\ &\Rightarrow (s_2, a, aZ)\end{aligned}$$

Example cont'd — Rejection



PDA execution.

$$\begin{aligned}(s_0, aaba, Z) &\Rightarrow (s_1, aba, aZ)\\ &\Rightarrow (s_1, ba, aaZ)\\ &\Rightarrow (s_2, a, aZ)\\ &\Rightarrow \text{???}\end{aligned}$$

Non-accepting execution.

- No transition possible, stuck without reaching final state
- rejection happens when transition function is undefined for current configuration (state, input, top of stack) or when word is consumed, and no epsilon transitions can bring us to a final state.

Example: Palindromes with ‘Centre Mark’

Example Language.

$$L = \{wcw^R \mid w \in \{a,b\}^* \land w^R \text{ is } w \text{ reversed}\}$$

Deterministic PDA that accepts L

Example: Palindromes with 'Centre Mark'

Example Language.

$$L = \{wcw^R \mid w \in \{a, b\}^* \land w^R \text{ is } w \text{ reversed}\}$$

Deterministic PDA that accepts L

- Push a's and b's onto the stack as we seem them;
- When we see c, *change state*;
- Now try to match the tokens we are reading with the tokens on top of the stack, popping as we go;
- If the top of the stack is the empty stack symbol Z, enter the final state via an ϵ-transition. Hopefully our input has been used up too!

Exercise. Define this formally!

Non-Deterministic PDAs

Deterministic PDAs

- transitions are a partial function

$$\delta \; : \; S \times (\Sigma \cup \{\epsilon\}) \times \Gamma \quad \nrightarrow \quad S \times \Gamma^*$$

$$\delta \; : \; (\text{state}, \text{input token or } \epsilon, \text{top-of-stack}) \quad \nrightarrow \quad (\text{new state}, \text{new top of stack})$$

- side condition about ϵ-transitions

Non-Deterministic PDAs

- transitions given by *relation*

$$\delta \; \subseteq \; S \times (\Sigma \cup \{\epsilon\}) \times \Gamma \quad \times \quad S \times \Gamma^*$$

- no side condition (at all).

Main differences

- for deterministic PDA: *at most* one transition possible
- for non-deterministic PDA: *zero or more* transitions possible

Non-Deterministic PDAs cont'd

Finite Automata

- non-determinism is *convenient*
- but doesn't give extra power (subset construction)
- can convert every NFA to an equivalent DFA

Push-down Automata.

- non-determinism *gives* extra power
- cannot convert *every* non-deterministic PDA to deterministic PDA
- there are context- free languages that can *only* be recognised by non-deterministic PDA
- intuition: non-determinism allows "guessing"

Grammar / Automata correspondence

- non-deterministic PDAs are more important
- they correspond to context-free languages

Example: Even-Length Palindromes

Palindromes of even length, *without* centre-marks

$$L = \{ww^R \mid w \in \{a,b\}^* \wedge w^R \text{ is } w \text{ reversed}\}$$

- this is a context-free language
- *cannot* be recognised by deterministic PDA
- intuitive reason: no centre-mark, so don't know when first half of word is read

Non-deterministic PDA for L has the transition

$$\delta(s, \epsilon, \gamma) \quad = \quad r/x$$

- $x \in \{a, b, Z\}$, s is the 'push' state and r the 'match and pop' state.

Intuition

- "guess" (non-deterministically) whether we need to enter "match-and-pop" state
- automaton gets stuck if guess is not correct (no harm done)
- automaton accepts if guess is correct

Grammars and PDAs

Theorem. Context-free languages and *non-deterministic* PDAs are equivalent

- for every CFL *L* there exists a PDA that accepts *L*
- if *L* is accepted by a non-deterministic PDA, then *L* is a CFL.

Proof. We only do one direction: construct PDA from CFL (i.e., CFL to PDA).

- other direction (i.e., PDA to CFL) quite complex.
- for our proof, since we have a CFL, by definition, there is CFG.

From CFG to PDA

Given. Context-Free Grammar $G = (V_t, V_n, S, P)$

Construct non-deterministic PDA $A = (Q, q_0, F, \Sigma, \Gamma, Z, \delta)$

States. q_0 (initial state), q_1 (working state) and $\underline{q_2}$ (final state).

Alphabet. $\Sigma = V_t$, terminal symbols of the grammar

Stack Alphabet. $\Gamma = V_t \cup V_n \cup \{Z\}$

Initialisation.

- push start symbol S onto stack, enter working state q_1
- $\delta(q_0, \epsilon, Z) \mapsto q_1/SZ$

Termination.

- if the stack is empty (i.e., just contains Z), terminate
- $\delta(q_1, \epsilon, Z) \mapsto \underline{q_2}/\epsilon$

From CFGs to PDAs: working state

Idea.

- build the derivation on the stack by expanding non-terminals according to productions
- if a terminal appears that matches the input, pop it
- terminate, if the entire input has been consumed

Expand Non-Terminals.

- non-terminals on the stack are replaced by right hand side of productions
- $\delta(q_1, \epsilon, A) \mapsto q_1/\alpha$ for all productions $A \to \alpha$

Pop Terminals.

- terminals on the stack are popped if they match the input
- $\delta(q_1, x, x) \mapsto q_1/\epsilon$ for all terminals x

Result of Construction. *Non-deterministic* PDA

- may have more than one production for a non-terminal

Example — Derive a PDA for a CFG

Arithmetic Expressions as a grammar:

$$\begin{array}{rcl} S & \to & S + T \mid T \\ T & \to & T * U \mid U \\ U & \to & (S) \mid \textbf{int} \end{array}$$

1. Initialise:

$$\delta(q_0, \epsilon, Z) \mapsto q_1/SZ$$

2. Expand non-terminals:

$$\begin{array}{rcl} \delta(q_1, \epsilon, S) & \mapsto & q_1/S + T \\ \delta(q_1, \epsilon, S) & \mapsto & q_1/T \\ \delta(q_1, \epsilon, T) & \mapsto & q_1/T * U \end{array} \qquad \begin{array}{rcl} \delta(q_1, \epsilon, T) & \mapsto & q_1/U \\ \delta(q_1, \epsilon, U) & \mapsto & q_1/(S) \\ \delta(q_1, \epsilon, U) & \mapsto & q_1/\textbf{int} \end{array}$$

CFG to PDA cont'd

3. Match and pop terminals:

$$\begin{aligned}
\delta(q_1, +, +) &\mapsto q_1/\epsilon \\
\delta(q_1, *, *) &\mapsto q_1/\epsilon \\
\delta(q_1, \textbf{int}, \textbf{int}) &\mapsto q_1/\epsilon \\
\delta(q_1, (, () &\mapsto q_1/\epsilon \\
\delta(q_1,),)) &\mapsto q_1/\epsilon
\end{aligned}$$

4. Terminate:

$$\delta(q_1, \epsilon, Z) \mapsto \underline{q_2}/\epsilon$$

Example Trace

$$
\begin{aligned}
(q_0,\ \textbf{int} * \textbf{int},\ Z) &\Rightarrow (q_1,\ \textbf{int} * \textbf{int},\ SZ)\\
&\Rightarrow (q_1,\ \textbf{int} * \textbf{int},\ TZ)\\
&\Rightarrow (q_1,\ \textbf{int} * \textbf{int},\ T * UZ)\\
&\Rightarrow (q_1,\ \textbf{int} * \textbf{int},\ U * UZ)\\
&\Rightarrow (q_1,\ \textbf{int} * \textbf{int},\ \textbf{int} * UZ)\\
&\Rightarrow (q_1,\ *\textbf{int},\ *UZ)\\
&\Rightarrow (q_1,\ \textbf{int},\ UZ)\\
&\Rightarrow (q_1,\ \textbf{int},\ \textbf{int}Z)\\
&\Rightarrow (q_1,\ \epsilon,\ Z)\\
&\Rightarrow (q_2,\ \epsilon,\ \epsilon)\\
&\Rightarrow \quad \textit{accept}
\end{aligned}
$$

Summary about PDAs

- Definition of deterministic PDA
- Definition of non-deterministic PDA
- PDA configurations
- Relation of PDAs to CFGs/CFLs (same!)
- Compilation: CFGs to PDAs