COMP3630 / COMP6363

week 1: **Finite Automata**

This Lecture Covers Chapter 2 of HMU: Finite Automata

slides created by: Dirk Pattinson, based on material by
Peter Hoefner and Rob van Glabbeck; with improvements by Pascal Bercher

convenor & lecturer: Pascal Bercher

The Australian National University

Semester 1, 2025

COMP3630/6363: Theory of Computation

Textbook. *Introduction to Automata Theory, Languages and Computation* by John E. Hopcroft, Rajeev Motwani, and Jeffrey D. Ullman [HMU].

Prerequisites. Chapter 1 of HMU (sets, functions, relations, induction)
(if you prefer lectures over reading, I uploaded one on YouTube)

Assessment.
- 5 assignments — each @ 10%
- 1 final exam — @ 50%

Labs. Participation is voluntary, but highly recommended.
- Wednesday, 9 am to 11 am and 11 am to 1 pm
- each tutorial covers the content of the same week
- tutor of tutorials: Eric Hall
- another (marking) tutor: Timothy Horscroft

Content. Languages / Automata / Computability / Complexity
This course is basically an advanced Math course.

Convenor. Pascal (Bercher), pascal.bercher@anu.edu.au

Lecturer. same!

Slides. Most of them mostly Dirk Pattinson

CSS

CLASS REPRESENTATIVES

Class Student Representation is an important component of the teaching and learning quality assurance and quality improvement processes within the ANU College of Systems and Society (CSS).

Each semester, we put out a call for Class Representatives for all ANU College of Systems and Society (CSS) courses. Students can nominate themselves for one or more of the courses they are enrolled in.

Australian National University

Roles and responsibilities:

The role of Student Representatives is to provide ongoing constructive feedback on behalf of the student cohort to Course Conveners and to Associate Directors (Education) for continuous improvements to the course.

- Act as the official liaison between your peers and convener.
- Be available and proactive in gathering feedback from your classmates.
- Attend regular meetings, and provide reports on course feedback to your course convener
- Close the feedback loop by reporting back to the class the outcomes of your meetings.

Note: Class representatives will need to be comfortable with their contact details being made available via Wattle to all students in the class.

For more information regarding roles and responsibilities, contact:

ANUSA CSS representatives (sa.cecc@anu.edu.au).

Why become a class representative?

- **Ensure students have a voice** to their course convener, lecturer, tutors, and College.
- **Develop skills sought by employers**, including interpersonal, dispute resolution, leadership and communication skills.
- **Become empowered**. Play an active role in determining the direction of your education.
- **Become more aware of issues influencing your University** and current issues in higher education.
- **Course design and delivery.** Help shape the delivery of your current courses, as well as future improvements for following years.

Want to be a class representative?
Nominate today!

Please nominate yourself to your course convener by end of Week 2

Content of this Chapter

- Deterministic Finite Automata
- Nondeterministic Finite Automata
- NFA with ϵ-transitions
- An Equivalence among the above three.

(This was all covered in COMP1600)

Additional Reading: Chapter 2 of HMU.

Preliminary Concepts

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  - $\Sigma = \{0, 1\}$ (**binary** alphabet)
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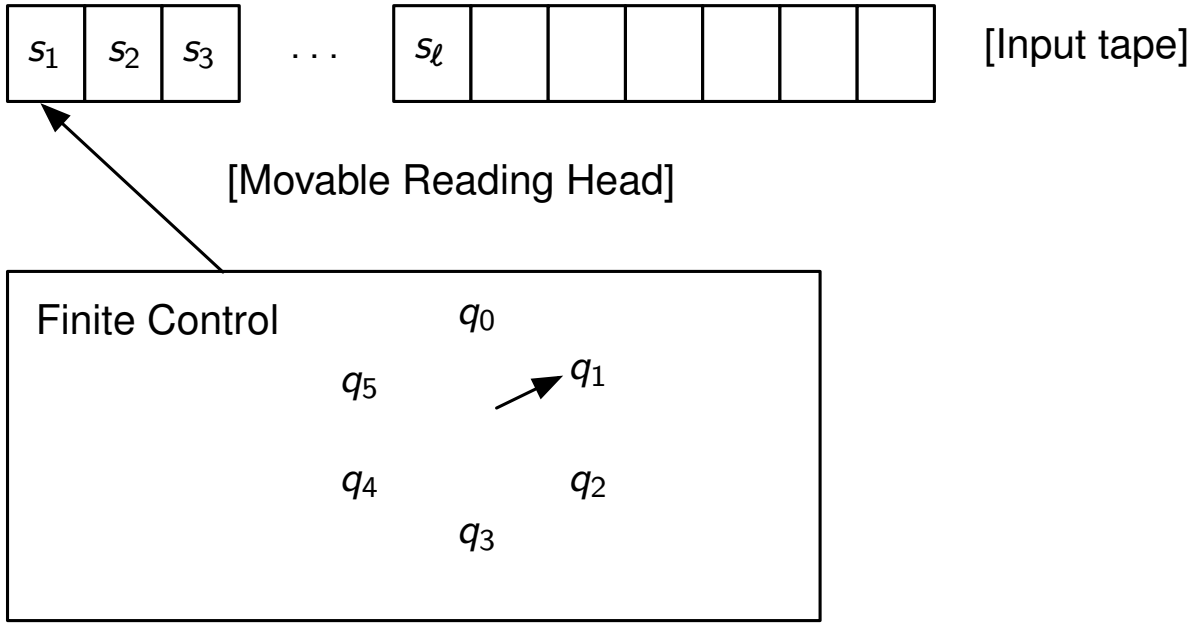
- A **(formal) language** is a subset of Σ^*.

The Deterministic Finite Automaton

Deterministic Finite Automaton (DFA)

Informally:



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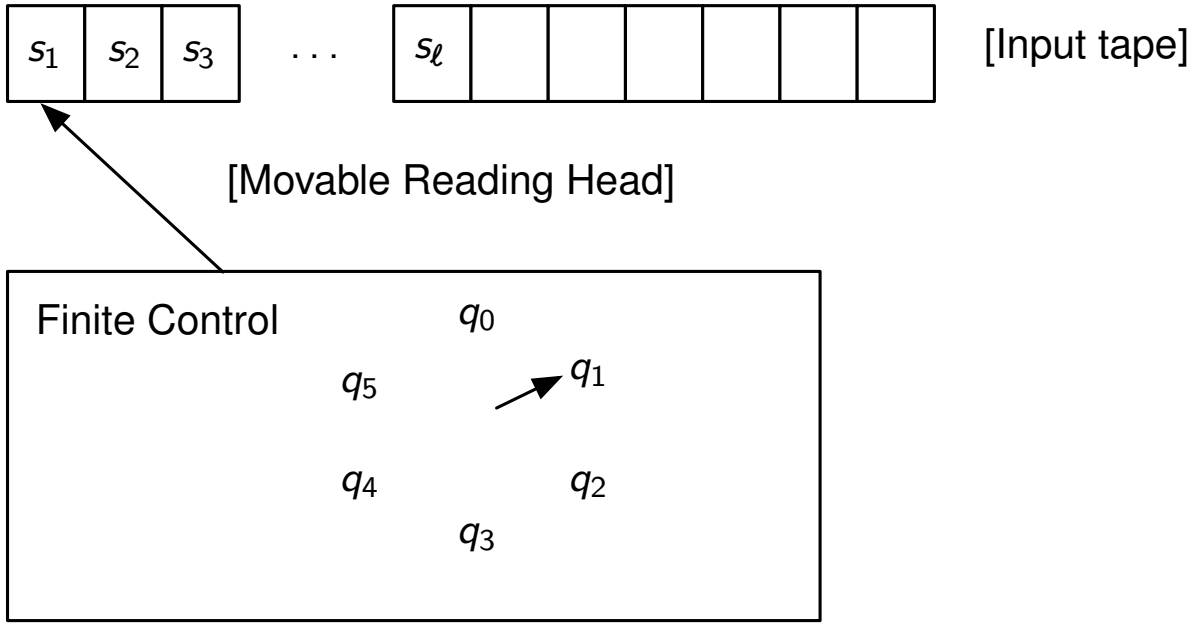

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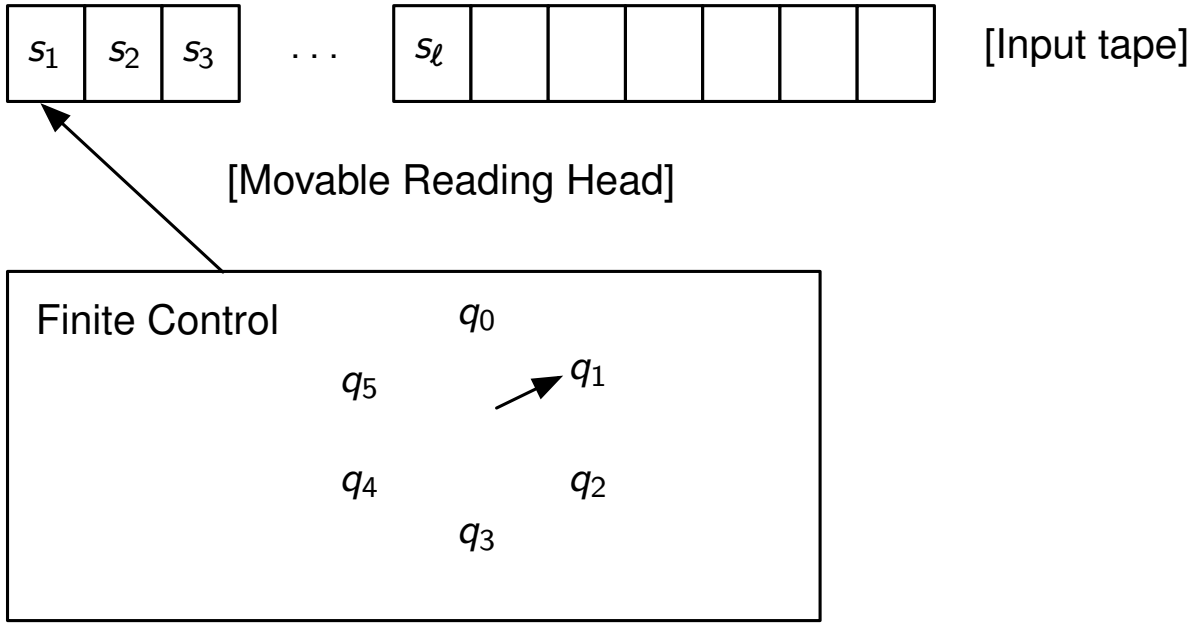

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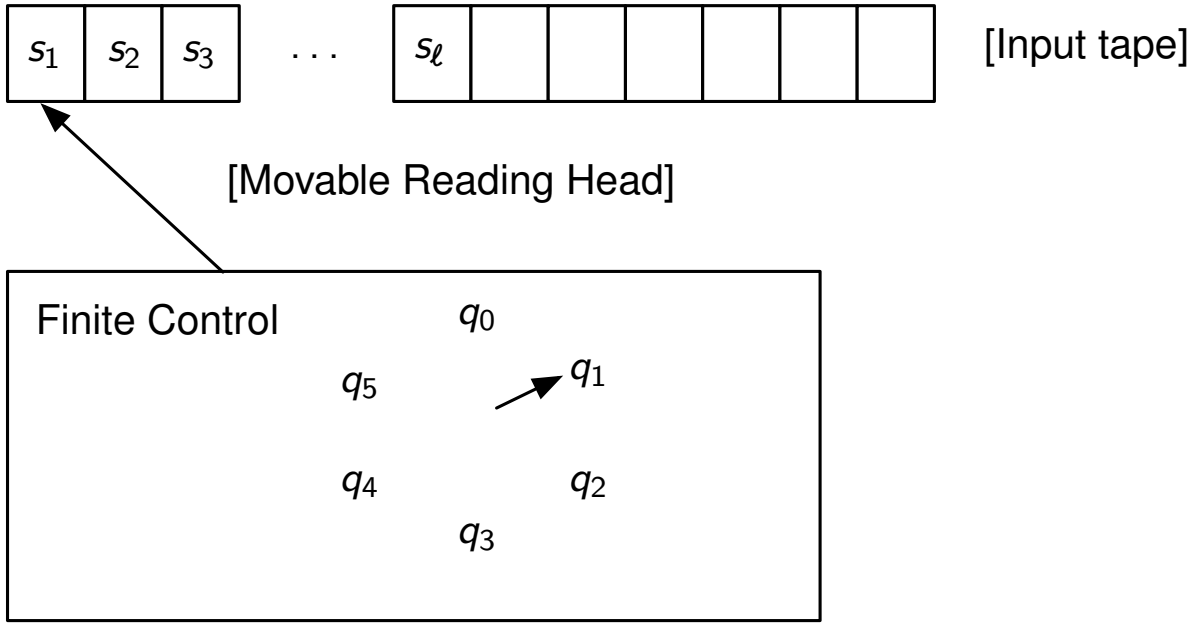

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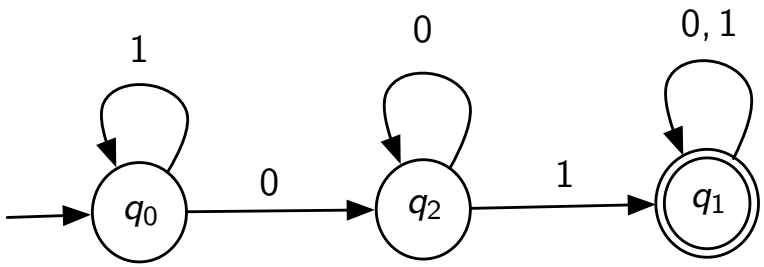
Transition Table:

	0	1
→ q_0	q_2	q_0
$*$ q_1	q_1	q_1
q_2	q_2	q_1

$F = \{q_1\}$

$\delta(q_0, 0) = q_2$
$\delta(q_0, 1) = q_0$

Transition Diagram:



Languages accepted by DFAs

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In other words:

- $\epsilon \in L(A) \Leftrightarrow q_0 \in F$
- For $k > 0$,

$$w = s_1 s_2 \cdots s_k \in L(A) \Leftrightarrow q_0 \xrightarrow{s_1} P_1 \xrightarrow{s_2} P_2 \xrightarrow{s_3} \cdots \xrightarrow{s_k} P_k \in F$$

An Example

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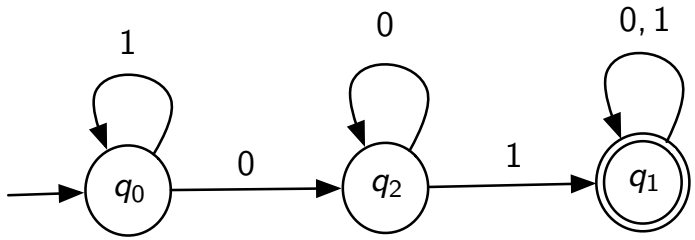

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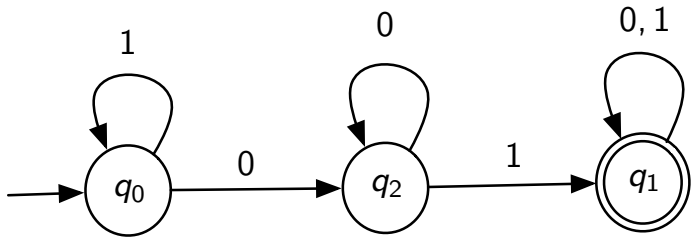

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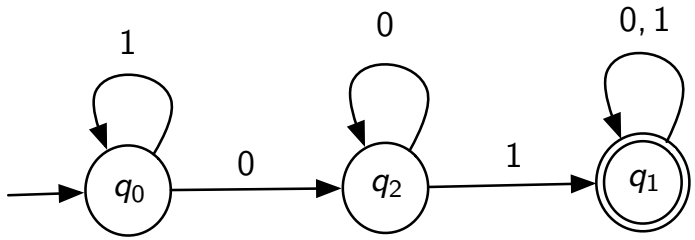

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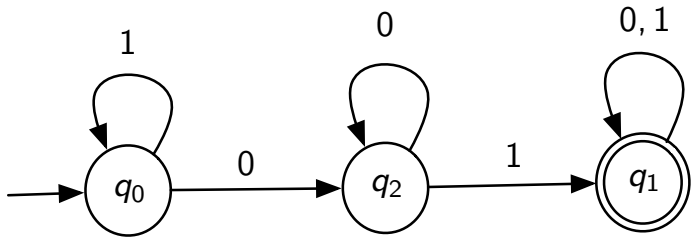

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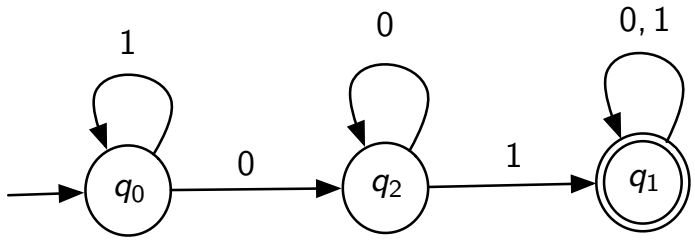

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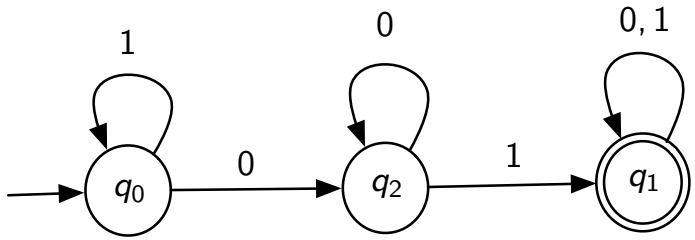

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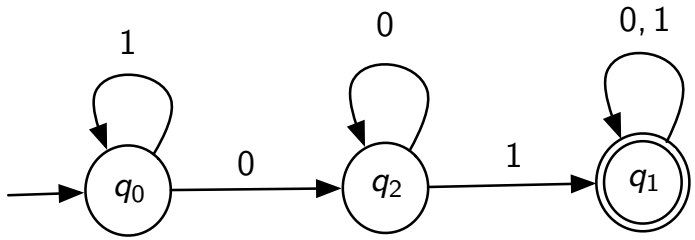

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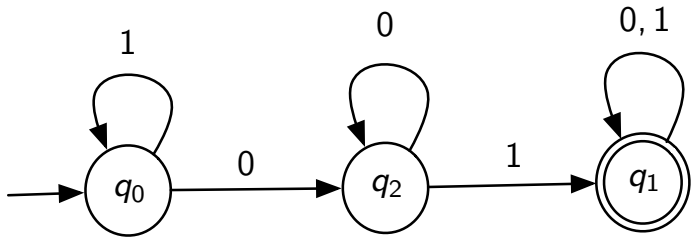

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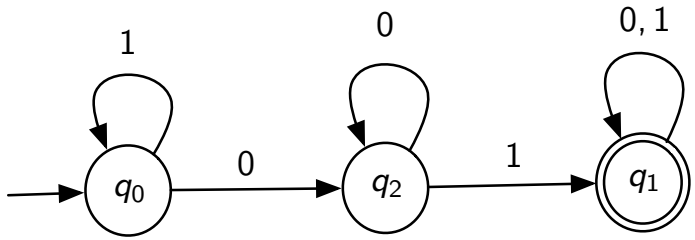

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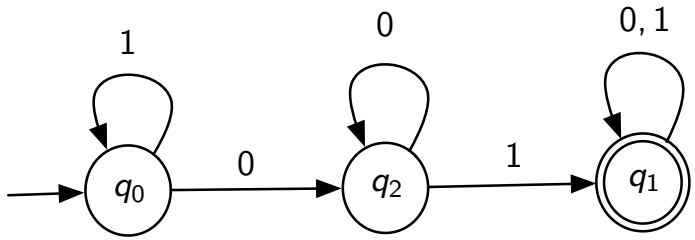

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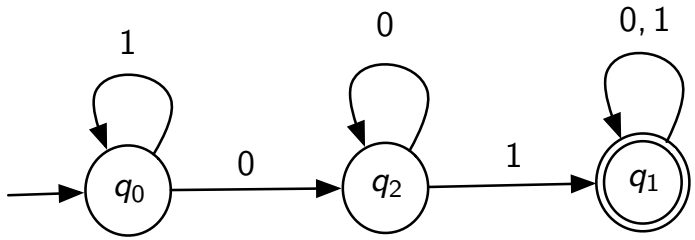

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- Remark 2: Multiple strings can have the same path of states. For example, 0010 and 0011 have the same sequence of states.

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  - DFAs have a finite number of states (and hence finite memory).
  - Given a DFA, there is always a long pattern it **cannot** 'remember' or 'track'
    - e.g., $L = \{0^n 1^n : n \in \mathbb{N}\}$ cannot be accepted by **any** DFA.

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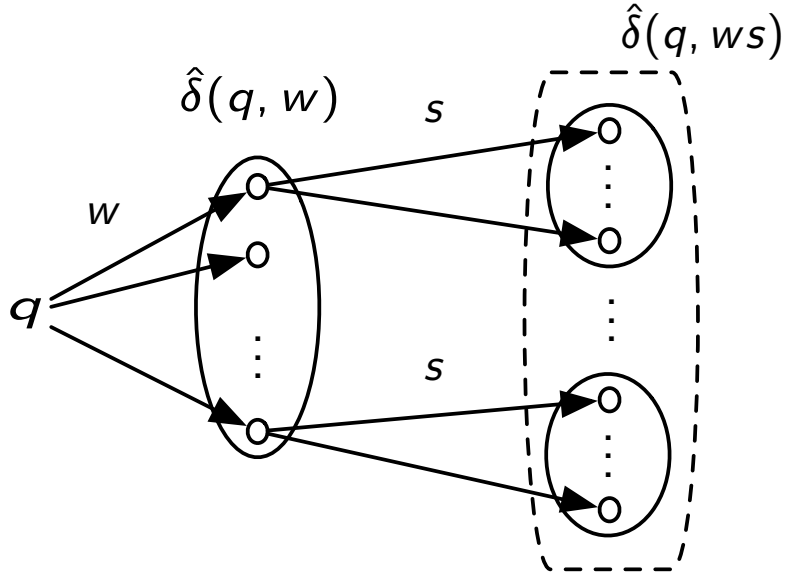

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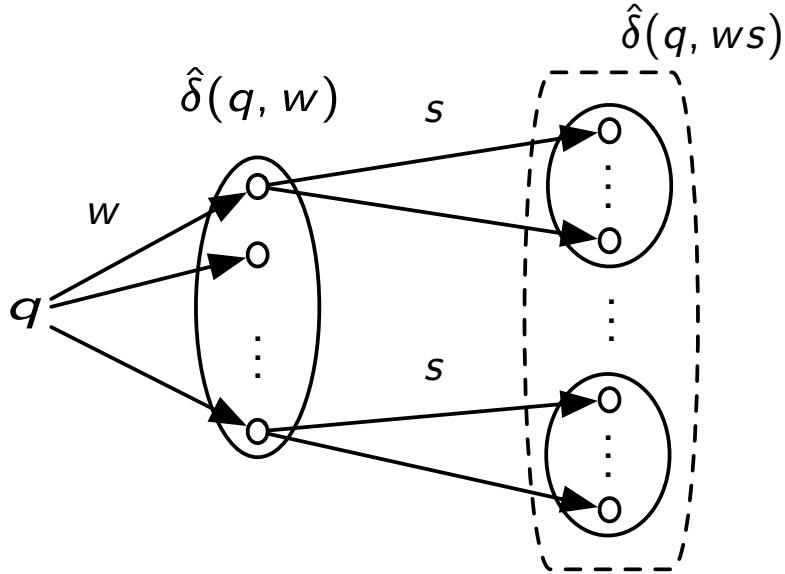

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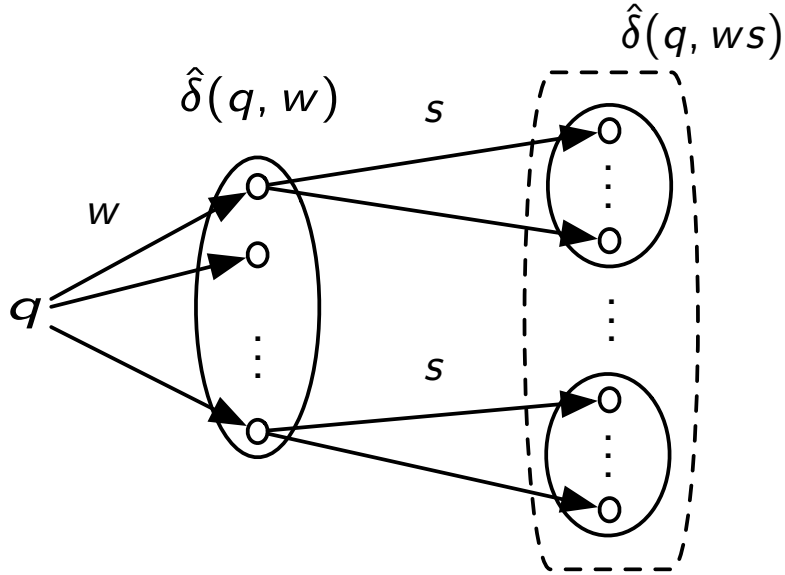

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In other words:

- $\epsilon \in L(A) \Leftrightarrow q_0 \in F$
- For $k > 0$,

$$w = s_1 s_2 \cdots s_k \in L(A) \Leftrightarrow \exists \text{ a path } q_0 \xrightarrow{s_1} P_1 \xrightarrow{s_2} P_2 \xrightarrow{s_3} \cdots \xrightarrow{s_k} P_k \in F$$

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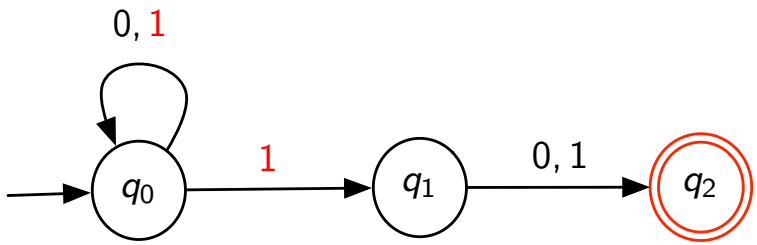

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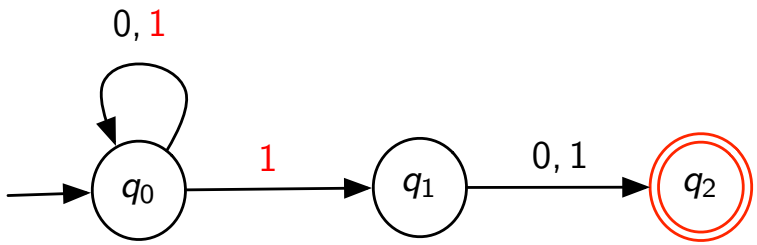

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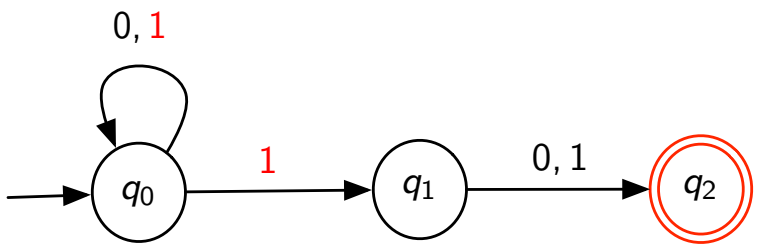

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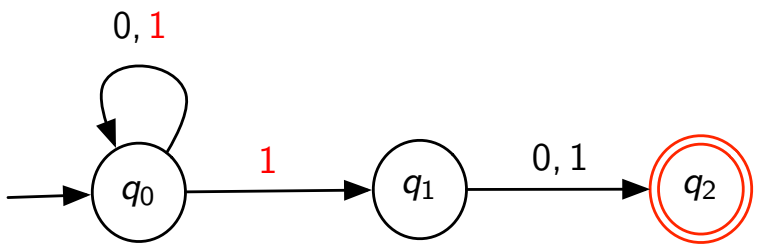

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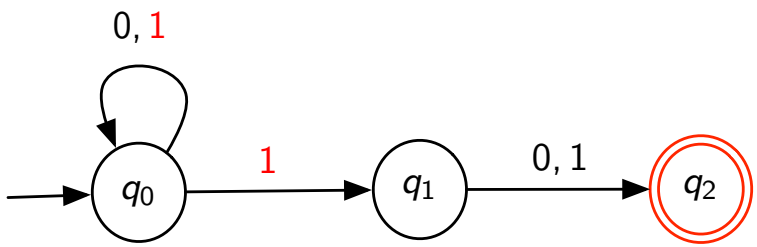

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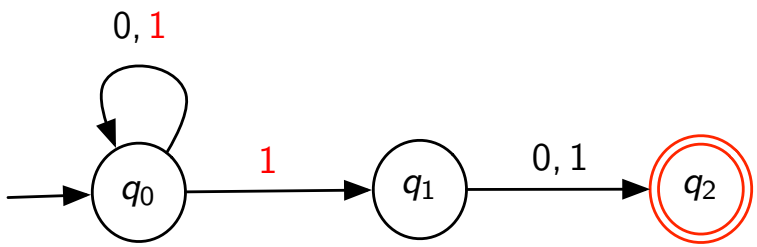

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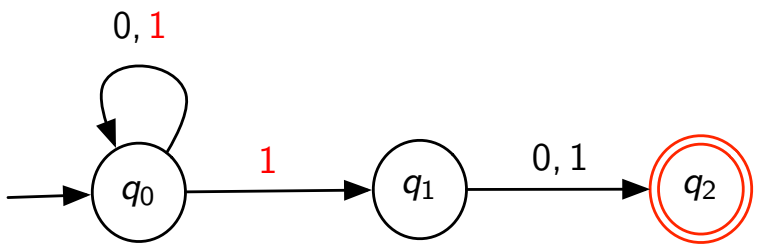

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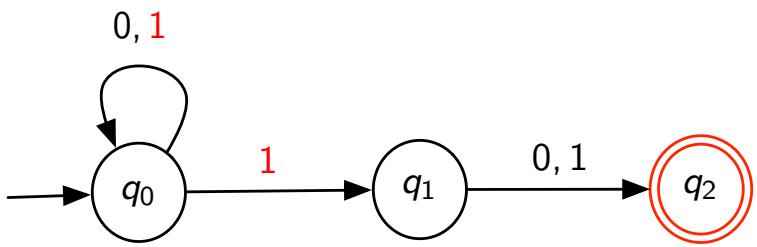

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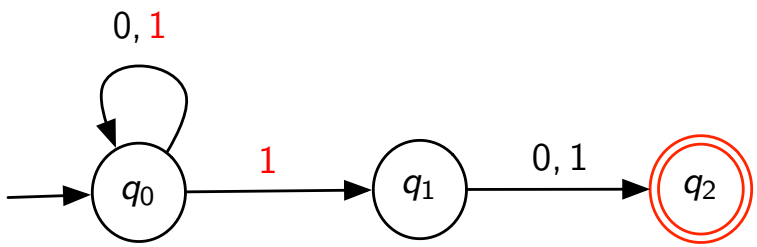

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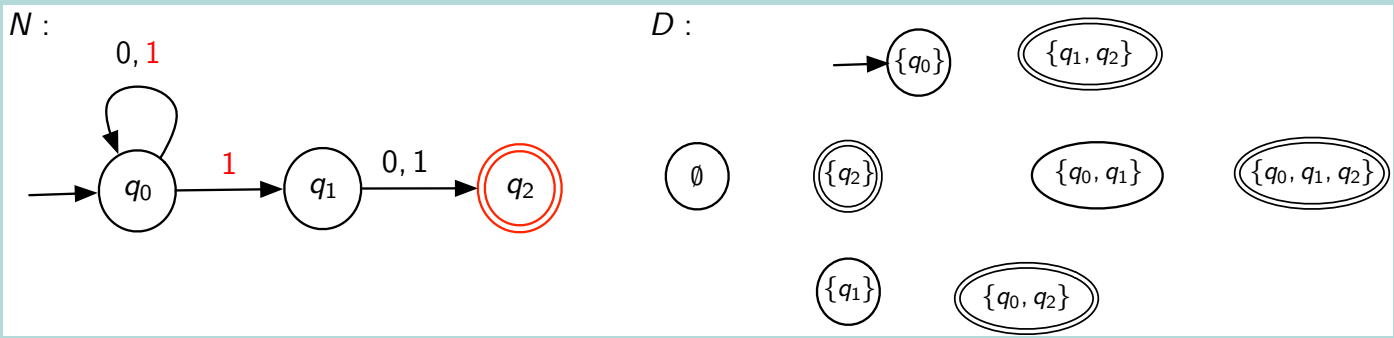

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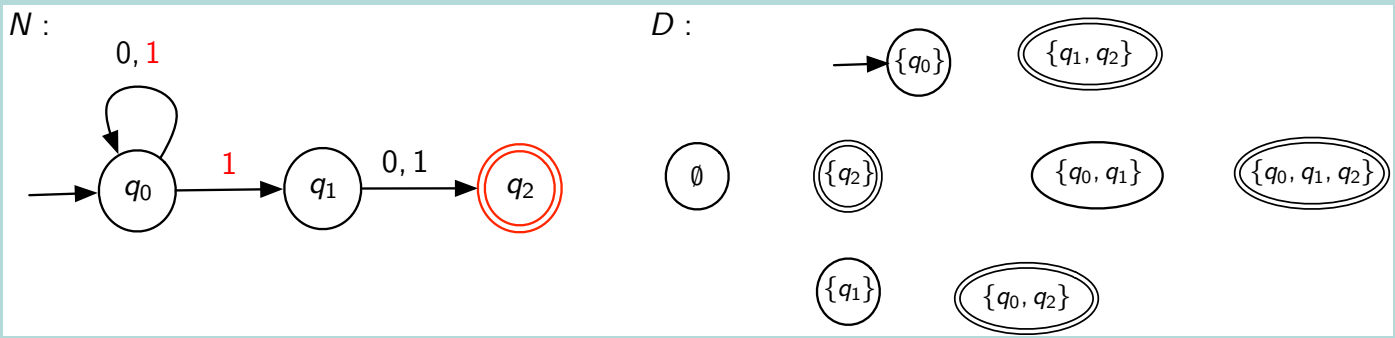

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- Hence, $\epsilon \in L(N) \Leftrightarrow q_0 \in F_N \Leftrightarrow \{q_0\} \in F_D \Leftrightarrow \epsilon \in L(D)$

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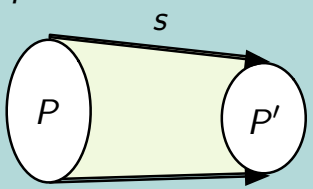
N: 

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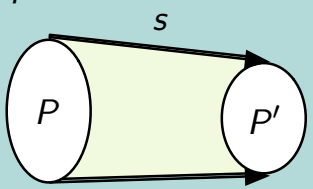
N: 

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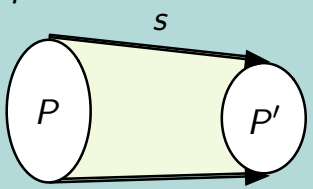
N: 

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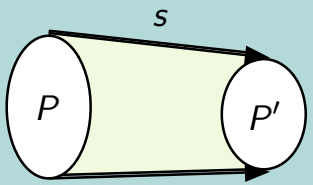
N: 

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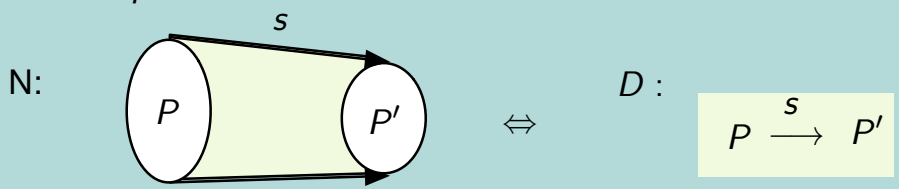

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- Thus, $\hat{\delta}_N(q_0, \cdot) = \hat{\delta}_D(\{q_0\}, \cdot)$, and hence the languages have to be identical.

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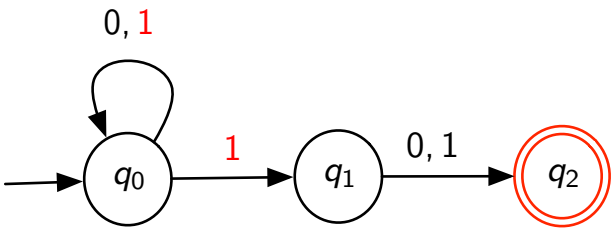

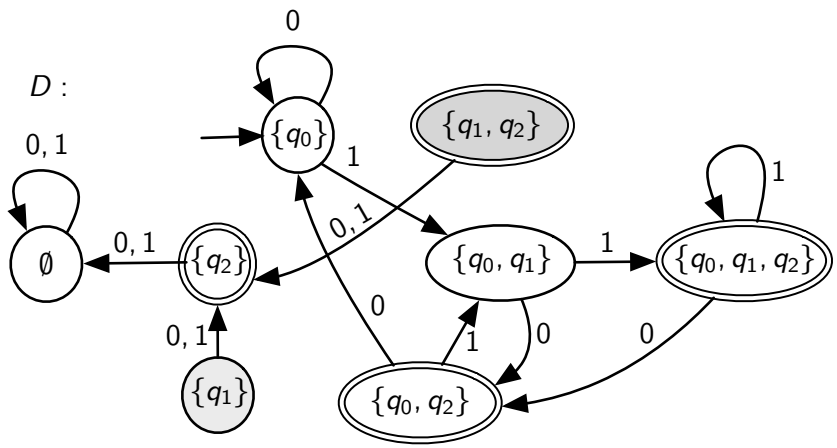

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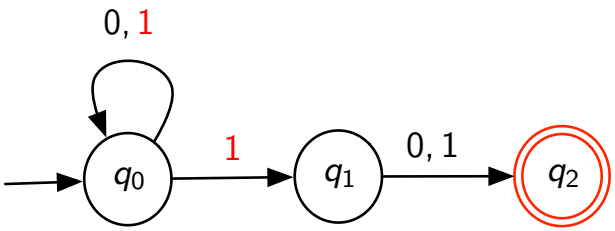

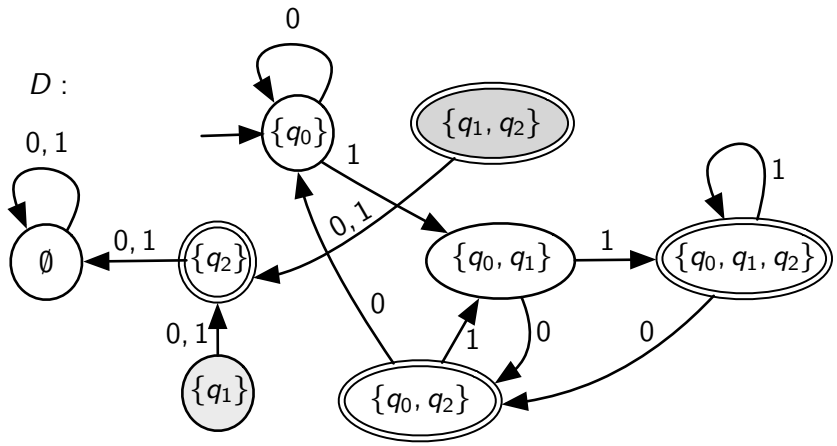

Example (from COMP1600)

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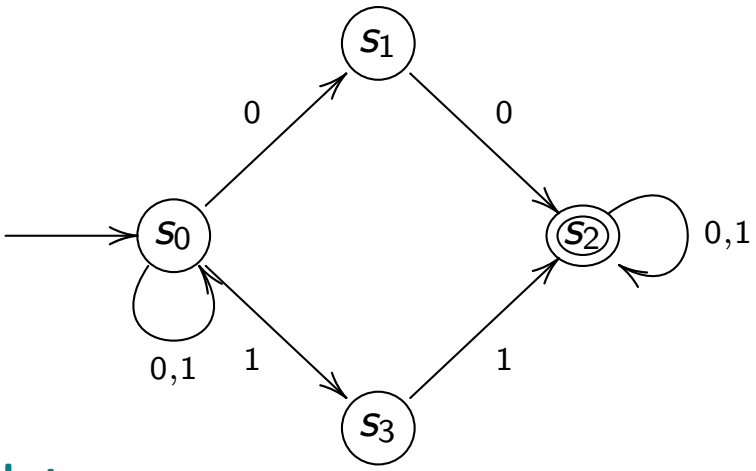

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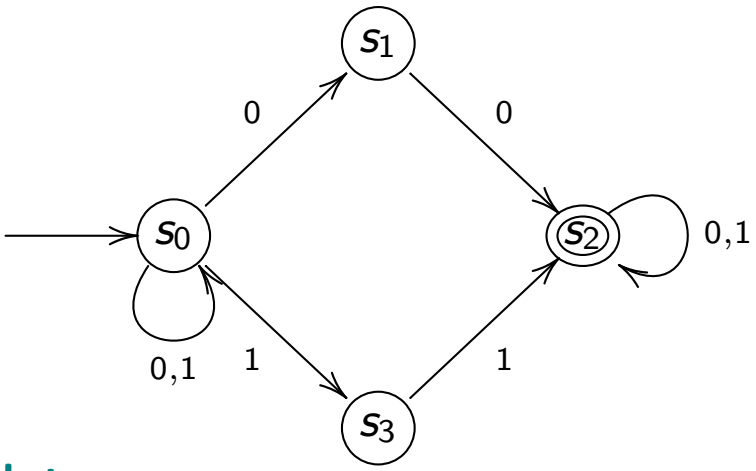

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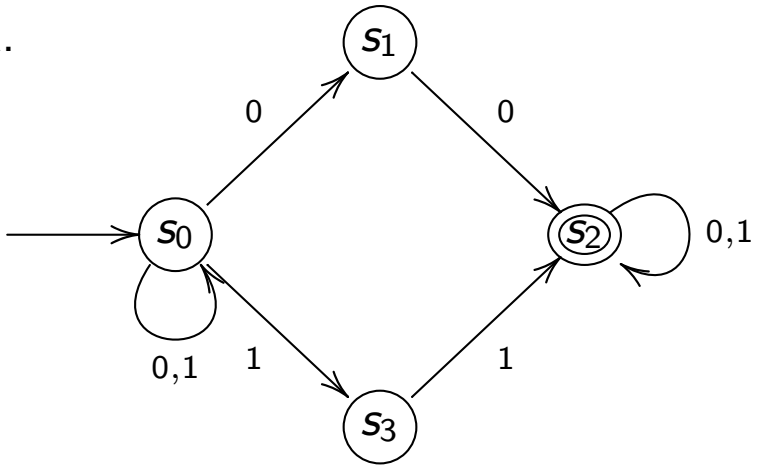
- Once the table is complete replace each DFA state set by a simple name

Determinisation Example, as Diagrams

Double Digits, as NFA.



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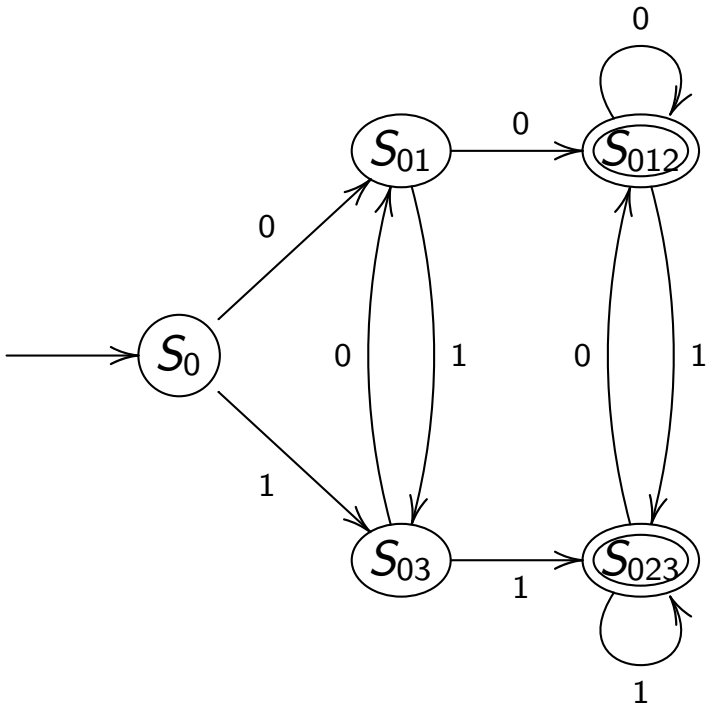

Transitions without Symbol Reading

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An ϵ-Nondeterministic Finite Automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ defined similar to a DFA with the exception of the transition function, which is defined to be:

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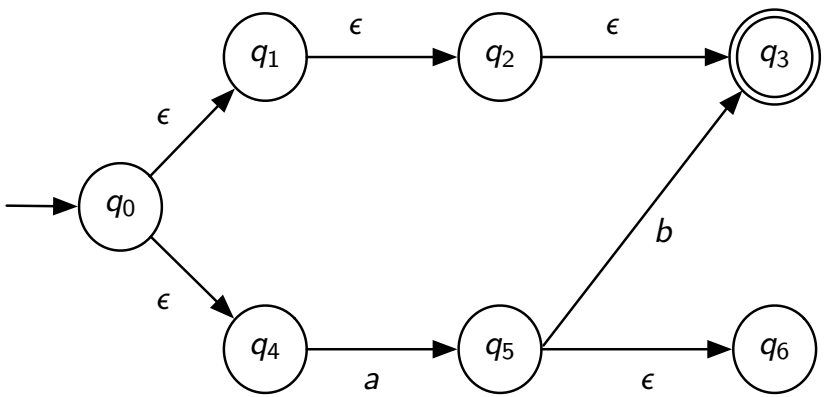

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q_1	$\{q_2\}$	$\emptyset$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\emptyset$
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q_4	$\emptyset$	$\{q_5\}$	$\emptyset$
q_5	$\{q_6\}$	$\emptyset$	$\{q_3\}$
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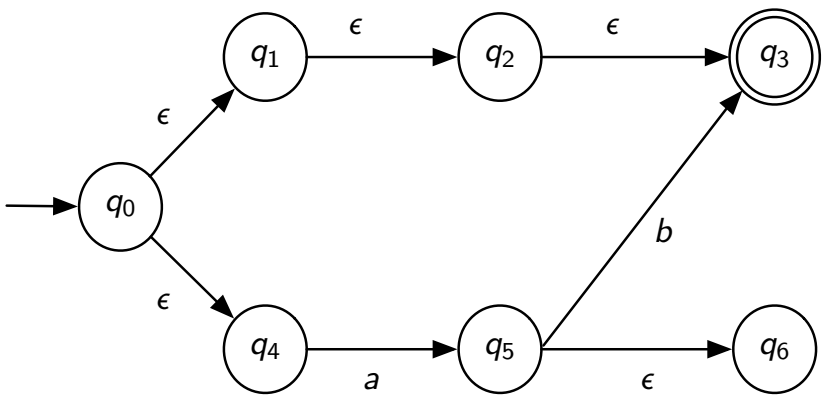

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q_6	$\emptyset$	$\emptyset$	$\emptyset$

› Without reading any input symbols, the state of the ϵ-NFA can transition:

From q_0 to q_1, q_4, q_2, or q_3. From q_1 to q_2, or q_3.

From q_2 to q_3. From q_5 to q_6.

Language Accepted by an ϵ-NFA

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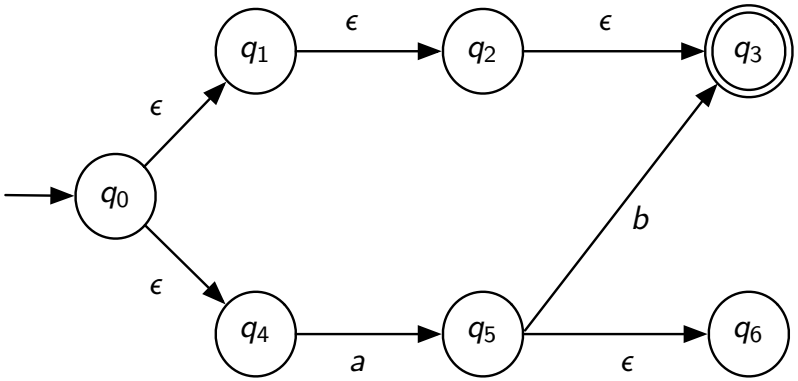

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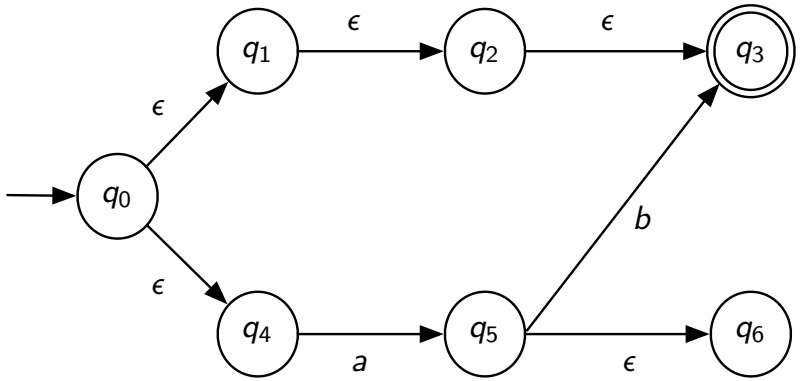

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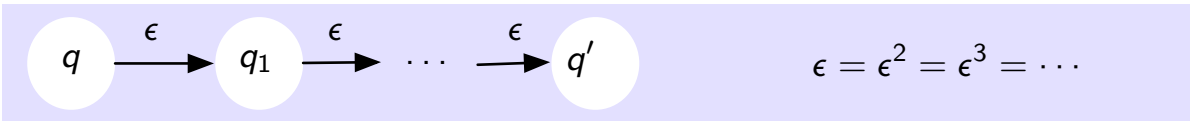

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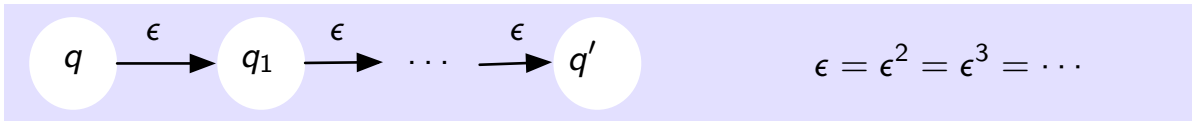

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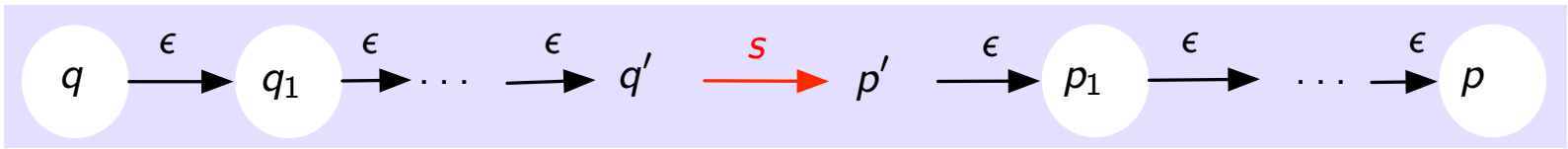

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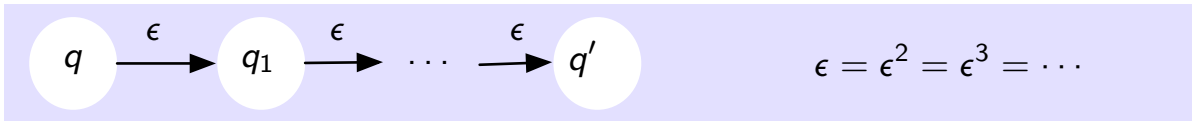

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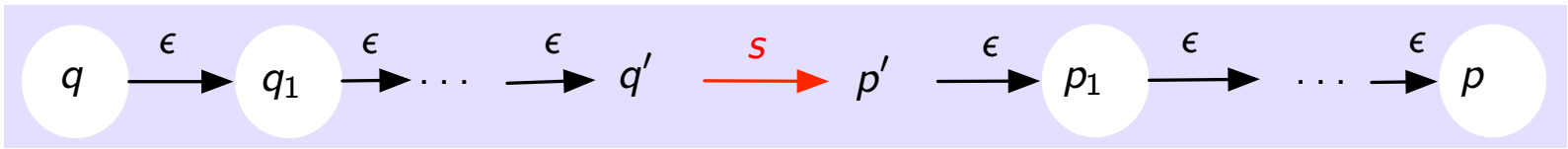

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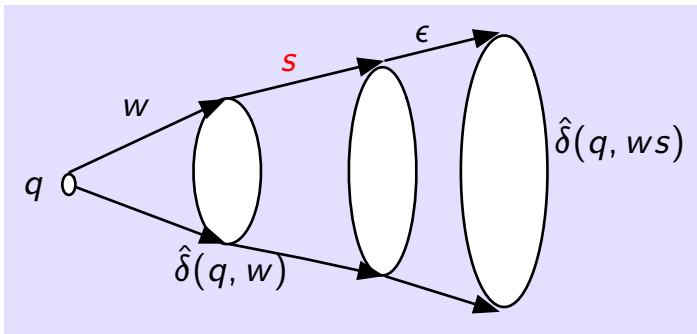

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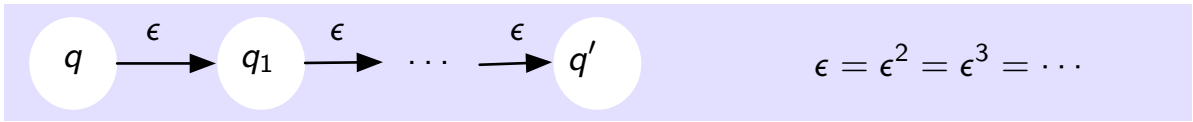

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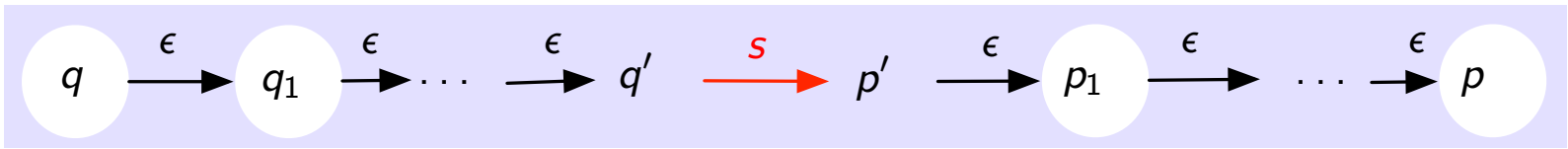

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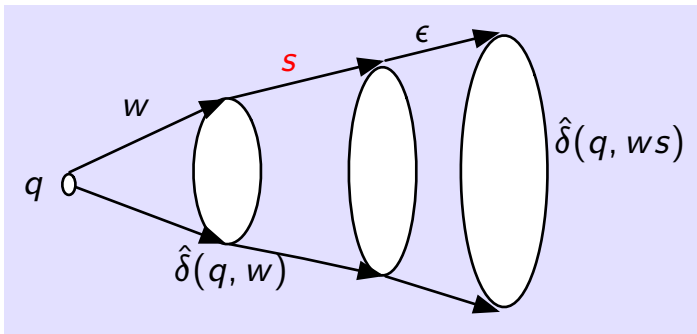

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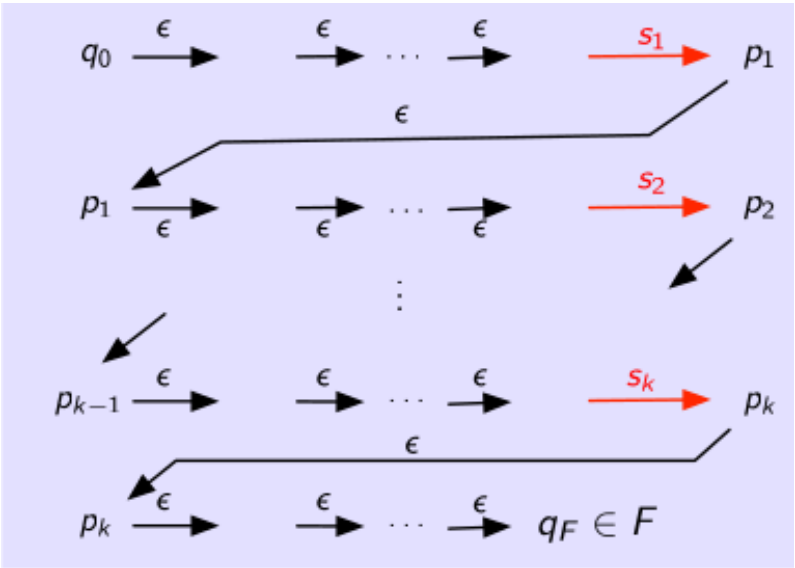
$$q_0 \xrightarrow{\epsilon} p_1 \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} p_r \in F$$

  - For $k > 0$, $w = s_1 s_2 \ldots s_k \in L(N) \Leftrightarrow \exists$ a path such as the following:



Do ϵ-NFAs Accept More Languages?

Theorem 2.5.1

Every Language L that is accepted by an ϵ-NFA is also accepted by some DFA.

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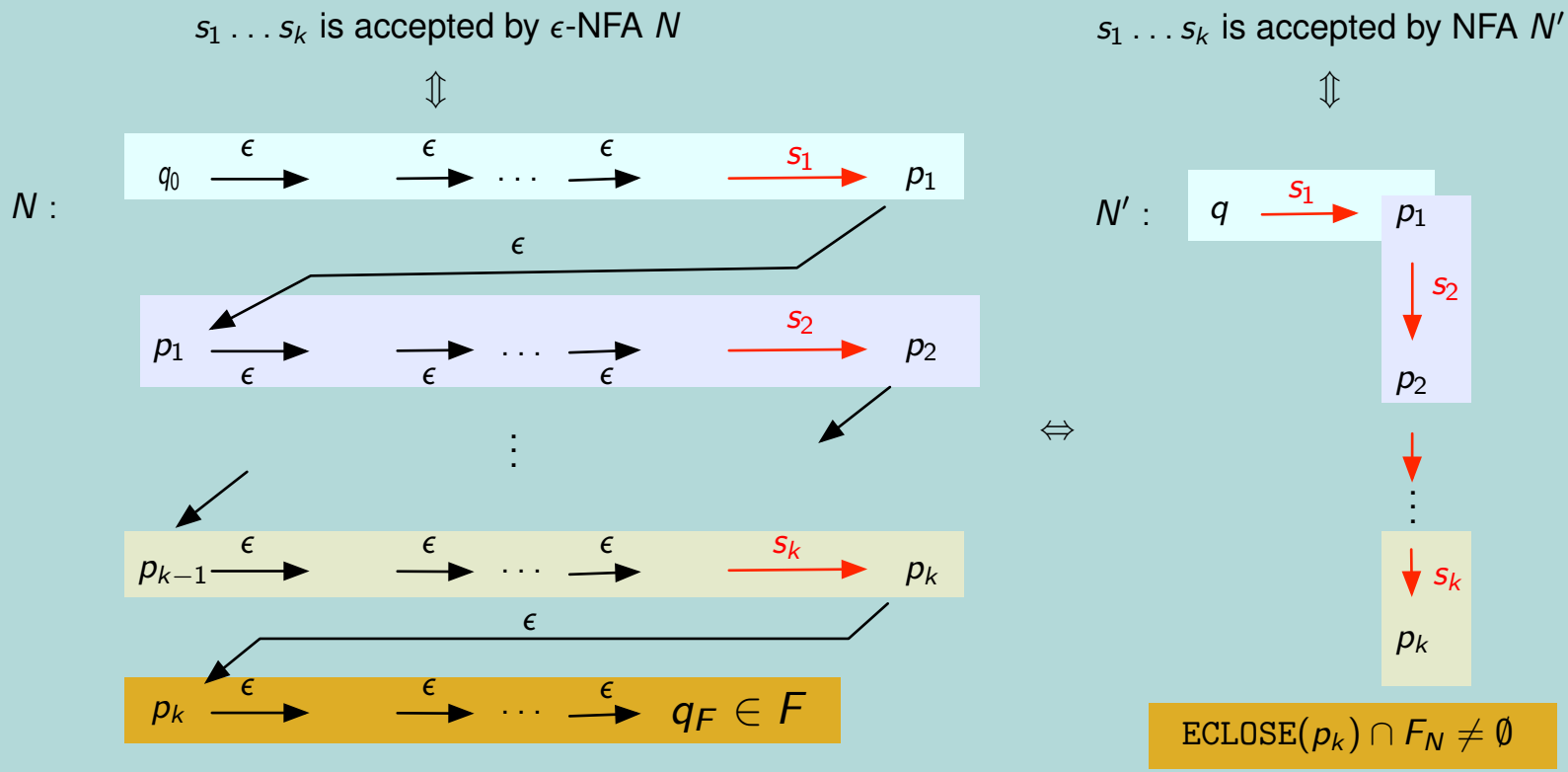
Proof of Theorem 2.5.1 (Abstract idea)



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$$Q_{N'} = Q_N \qquad q_0' = q_0 \qquad F_N' = \{q \in Q_N : \texttt{ECLOSE}(q) \cap F_N \neq \emptyset\}$$

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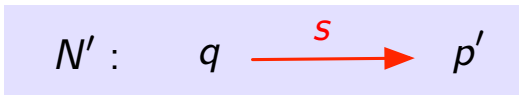
N : q can transition to p' after a few ϵ-transitions, and a single read of $s \in \Sigma$.

⇕



N': q can transition to p' after reading s.

Summary

To Summarize...

$$\text{Languages accepted by DFAs} = \text{Languages accepted by NFAs} = \text{Languages accepted by } \epsilon\text{-NFAs}$$

› Allowing non-determinism and/or ϵ-transitions does not improve the language acceptance power of (finite) automata.