COMP3630 / COMP6363

# *week 1:* **Regular Expressions and Languages**

This Lecture Covers Chapter 3 of HMU: Regular Expressions and Languages

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*convenor & lecturer:* Pascal Bercher

**The Australian National University**

Semester 1, 2025

# Content of this Chapter

- Introduction to regular expressions and regular languages
- Equivalence of classes of regular languages and languages accepted
- Algebraic laws of (abstract) regular expressions

Additional Reading: Chapter 3 of HMU.

# Regular Expressions and Languages

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- Each regular expression $r$ denotes a language $L(r) \subseteq \Sigma^*$

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- **Remark**: Some authors/texts use $|$ instead of $+$. HMU uses $+$.
- Precedence Rules:

$$(\cdot) > * > \cdot > +$$

where $>$ is 'binds more strongly than', and both $+$ and $\cdot$ associate to the left.

# Regular Expressions: Examples

- $r = 0 + 11^*10$ is a regular expression
  - with brackets that indicate precedence: $r = 0 + (1(1^*)10)$
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- Computing the language:

$$\begin{aligned} L(r) &= L(0) \cup L(11^*10) \\ &= \{0\} \cup L(1) \cdot L(1^*) \cdot L(1) \cdot L(0) \\ &= \{0\} \cup \{1\} \cdot \{1\}^* \cdot \{1\} \cdot \{0\} \\ &= \{0\} \cup \{1\} \cdot \{1^n \mid n \geq 0\} \cdot \{1\} \cdot \{0\} \\ &= \{1^i 0 \mid i \neq 1\} \end{aligned}$$

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# DFAs and Regular Languages

# Regular Languages: Some Basic Properties

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  Proof by induction on the length of $w$. For $w = \epsilon$, $\{w\} = \{\epsilon\} = L(\epsilon)$. For $w$ of the form $w's$, we have (by induction) $r$ s.t. $\{w'\} = L(r)$ so that $\{w\} = \{w's\} = L(rs)$.

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- **Corollary 2:** Any finite language is regular.

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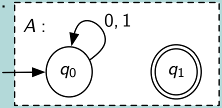
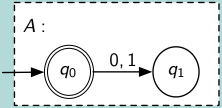



$\epsilon$

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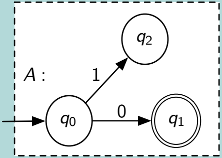


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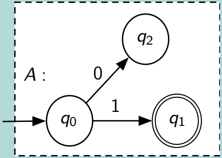

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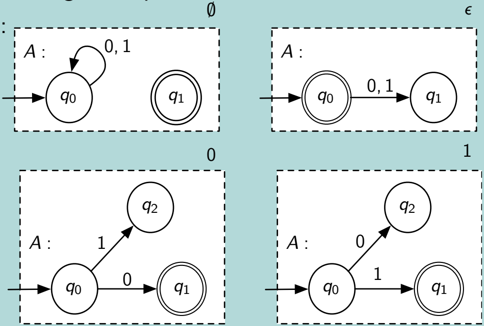


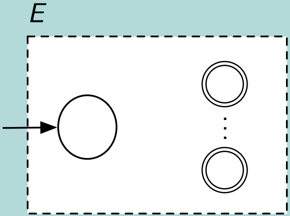

Note that these automata could be made smaller:

- $\emptyset/\epsilon$ only keep initial state and no transitions since runs with non-existent transitions fail.
- $0/1$ $q_2$ can be removed since runs with non-existent transitions fail.

# DFAs and Regular Languages

## Proof of Theorem 3.2.3 (Cont'd)

- Induction $E^*$:



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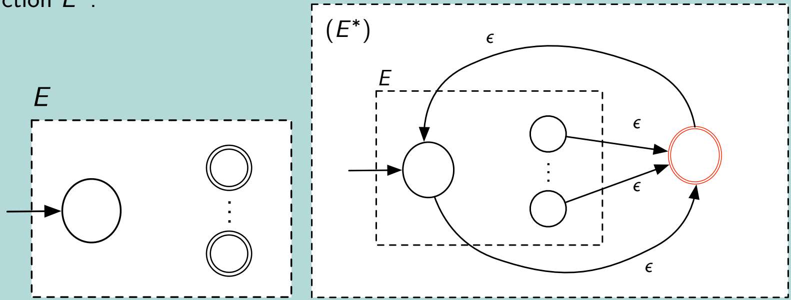

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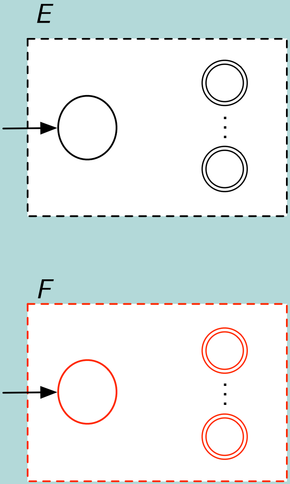

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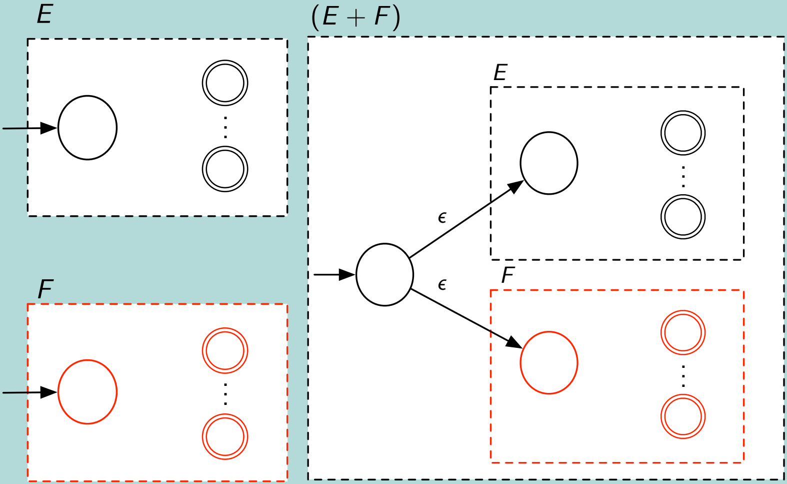

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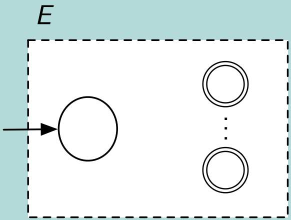


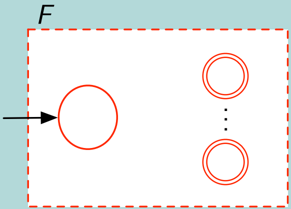

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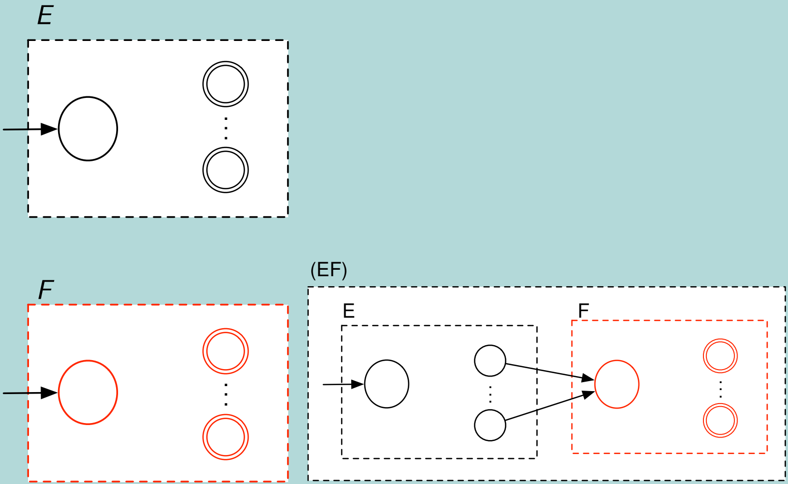

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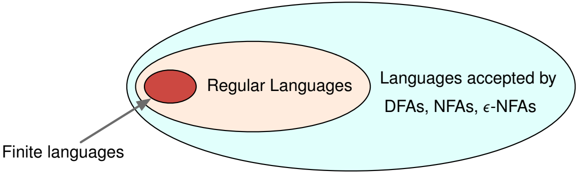
Regular Languages
Languages accepted by
DFAs, NFAs, ϵ-NFAs
Finite languages

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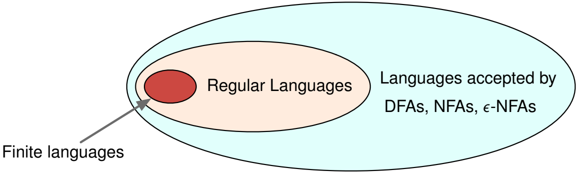


- Is the inclusion strict?
- Are there languages accepted by DFAs that are not regular?

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# DFAs and Regular Languages

**Theorem 3.2.4**

*For every DFA $A$, there is a regular expression $E$ such that $L(A) = L(E)$.*

**Proof of Theorem 3.2.4**

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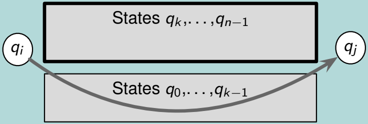

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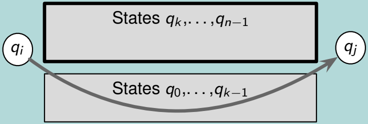


- Idea: prove that (a) each $R(i, j, k)$ is regular, and (b) $L(A)$ is a union of $R(i, j, k)$'s.

# DFAs and Regular Languages

## Proof of Theorem 3.2.4 (Cont'd)

- Note that $L(A) = \bigcup_{j:q_j \in F} R(0, j, n)$. (i.e., paths that start in $q_0$ and end in an accepting state with intermediate nodes $q_0, q_1, \ldots, q_{n-1}$ (all nodes))

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  $\Rightarrow$ $R(i, j, 0)$ is a regular language [Corollary 2]
- **Induction:** Let $R(i, j, \ell)$ be regular for $i, j \in \{0, \ldots, n-1\}$ and $0 \leq \ell < k$. Consider $R(i, j, k)$ for $i, j \in \{0, \ldots, n-1\}$.

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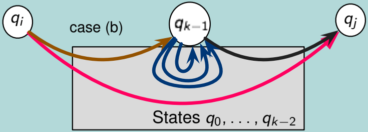

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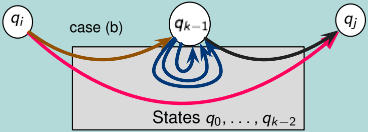


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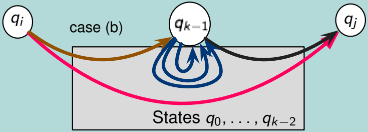


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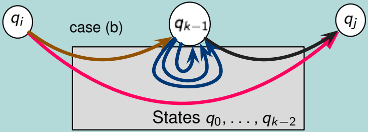


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- Hence, $R(i, j, k) = R(i, j, k-1) \cup \{\text{Case (b) strings}\}$.

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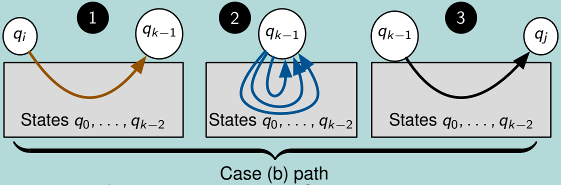


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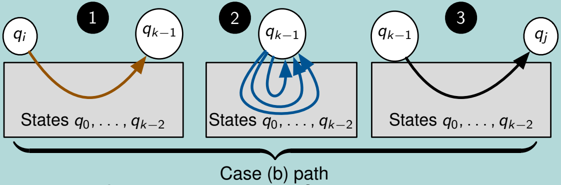


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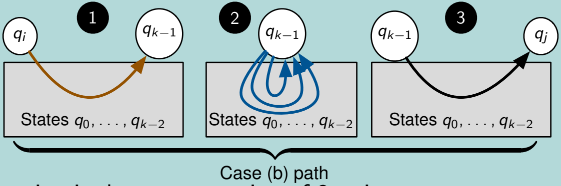


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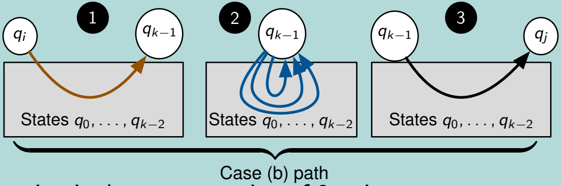


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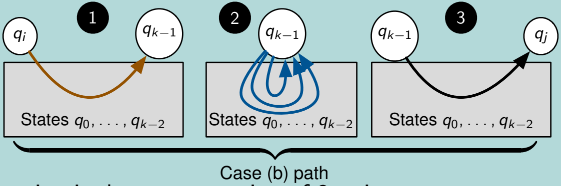


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  Thus,

$$R(i,j,k) = R(i,j,k-1) \cup [R(i,k-1,k-1)R(k-1,k-1,k-1)^* R(k-1,j,k-1)]$$

- From Thm 3.2.2, it follows that $R(i,j,k)$ is regular for any $i,j,k$. Thus, $L(A)$ is regular.

# Equivalence of Languages

- The following are indeed equivalent:
  - The class of regular languages
  - The class of languages accepted by DFAs
  - The class of languages accepted by NFAs
  - The class of languages accepted by $\epsilon$-NFAs

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- We can introduce a notion of equality of (abstract) regular expression:

Abstract regular expressions $E_1 = E_2 \Leftrightarrow$ For any assignment of languages to the variables contained in $E_1, E_2$, their evaluations equal (i.e., $L(E_1) = L(E_2)$)

# Algebraic Laws of Abstract Regular Expressions

- Commutativity: $\mathtt{L} + \mathtt{M} = \mathtt{M} + \mathtt{L}$ (Union is commutative)
  $\mathtt{LM} \neq \mathtt{ML}$ (Concatenation is not commutative)
- Associativity: $(\mathtt{L} + \mathtt{M}) + \mathtt{N} = \mathtt{L} + (\mathtt{M} + \mathtt{N})$ (Union is associative)
  $(\mathtt{LM})\mathtt{N} = \mathtt{L}(\mathtt{MN})$ (Concatenation is associative)
- Identity: $\emptyset + \mathtt{L} = \mathtt{L} + \emptyset = \mathtt{L}$ ($\emptyset$ is the identity element for $+$)
  $\epsilon\mathtt{L} = \mathtt{L}\epsilon = \mathtt{L}$ ($\epsilon$ is the identity element for concatenation)
- Annihilator: $\emptyset\mathtt{L} = \mathtt{L}\emptyset = \emptyset$
- Idempotent: $\mathtt{L} + \mathtt{L} = \mathtt{L}$
- Distributive: $\mathtt{L}(\mathtt{M} + \mathtt{N}) = \mathtt{LM} + \mathtt{LN}$
  $(\mathtt{M} + \mathtt{N})\mathtt{L} = \mathtt{ML} + \mathtt{NL}$
- Kleene $*$: $(\mathtt{L}^*)^* = \mathtt{L}^*$;   $\emptyset^* = \epsilon$;   $\epsilon^* = \epsilon$.

# Summary

# We can now summarize:

- We know what formal languages are.
- DFAs and all NFAs accept the same class of languages.
- Also regular expression accept exactly the same class of languages as DFAs/NFAs/$\epsilon$-NFAs.
- We saw some properties of regular languages (and will see more in the tutorials).
- We also saw abstract regular expressions.