COMP3630 / COMP6363

week 2: **Properties of Regular Languages**

This Lecture Covers Chapter 4 of HMU: Properties of Regular Languages

slides created by: Dirk Pattinson, based on material by
Peter Hoefner and Rob van Glabbeck; with improvements by Pascal Bercher

convenor & lecturer: Pascal Bercher

The Australian National University

Semester 1, 2025

Content of this Chapter

- Pumping Lemma for regular languages
- Some more properties of regular languages
- Decision properties of regular languages
- Equivalence and minimization of automata

Additional Reading: Chapter 4 of HMU.

Pumping Lemma for Regular Languages

Pumping Lemma

- We know: If a language is given by a regular expression, or a DFA, it is regular.
- What can we say if a language is defined by enumeration or by a predicate?
  - Is $L = \{w \in \{0,1\}^* : w \text{ does not contain } 10\}$ regular?
  - Is $L = \{0^n 1^n : n \geq 0\}$ regular?
  - How do we answer such questions without delving into each?
- Is there an inherent structure to the strings belonging to a regular language?

Lemma 4.1.1 (Pumping Lemma for Regular Languages)

Let L be a regular language. There there exists an $n \in \mathbb{N}, n \geq 1$ (depending on L) such that for any string $w \in L$ with $|w| \geq n$, there exist strings x, y, z such that:
(1) $w = xyz$
(2) $|xy| \leq n$
(3) $|y| > 0$
(4) $xy^i z \in L$ for all $i \in \mathbb{N}$

Proof of the Pumping Lemma

- Let DFA $A = (Q, \Sigma, \delta, q_0, F)$ accept L, and let $n := |Q|$.
- The claim is vacuously true if L contains only strings of length $n - 1$ or less.

Why?

Reminder:

Lemma 4.1.2 (Pumping Lemma for Regular Languages)

Let L be a regular language. There there exists an $n \in \mathbb{N}, n \geq 1$ (depending on L) such that for any string $w \in L$ with $|w| \geq n$, there exist strings x, y, z such that:
(1) $w = xyz$
(2) $|xy| \leq n$
(3) $|y| > 0$
(4) $xy^i z \in L$ for all $i \in \mathbb{N}$

Because the lemma requires: "$\exists n \in \mathbb{N}, n \geq 1$, s.t. $\forall w \in L$ with $|w| \geq n$ ..."
Thus, no such w exist, and therefore the lemma is (vacuously) true.

Proof of the Pumping Lemma

- Let DFA $A = (Q, \Sigma, \delta, q_0, F)$ accept L, and let $n := |Q|$.
- The claim is vacuously true if L contains only strings of length $n-1$ or less.
- Suppose L contains a string $w = s_1 \cdots s_k \in L$ with $|w| = k \geq n$.
- Then, there must be a sequence of transitions that move A from q_0 to some final state upon reading w.

$$\underbrace{q_0 = q_{i_0} \longrightarrow q_{i_1} \longrightarrow q_{i_2} \longrightarrow \cdots \longrightarrow q_{i_n}}_{n \text{ symbols and } n+1 \text{ states}} \longrightarrow \cdots \longrightarrow q_{i_k} \in F$$

- **SOME** state must be visited (at least) twice. Let $q_{i_a} = q_{i_b}$ for $i_0 \leq i_a < i_b \leq i_n$.

$$q_0 = q_{i_0} \overbrace{\xrightarrow{s_1} \cdots \xrightarrow{s_{i_a}}}^{x \text{ is read}} q_{i_a} \overbrace{\xrightarrow{s_{i_a+1}} \cdots \xrightarrow{s_{i_b}}}^{y \text{ is read}} q_{i_b} \overbrace{\xrightarrow{s_{i_b+1}} \cdots \xrightarrow{s_{i_n}} q_{i_n} \xrightarrow{s_{i_n+1}} \cdots \xrightarrow{s_{i_k}}}^{z \text{ is read}} q_{i_k} \in F$$

- $xy^iz \in L \forall i \in \mathbb{N}$ holds since the path for xy^iz is derived from the above either by deleting the subpath between q_{i_a} and q_{i_b} or by repeating it. All such paths end in $q_{i_k} \in F$.

Applications of the Pumping Lemma

Using the Pumping Lemma, we can show

- $L = \{0^n 1^n : n \geq 0\}$ is **not** regular.
  - Suppose it is. By the pumping lemma, there exists $k \geq 1$ such that any $w \in L$, $|w| \geq k$ can be split as $w = xyz, |y| \geq 1$ and $|xy| \leq k$ s.t. $xy^i z \in L$ for all $i \geq 0$.
  - let's apply this to the string $w = 0^k 1^k \in L$:

$$\overbrace{\underbrace{0\ldots0}_{x}\underbrace{0\ldots0}_{y}\underbrace{0\ldots0}_{z}}^{k}\overbrace{1\ldots1}^{k}$$

  - As $|xy| \leq k$, this means that $x = 0^i$ and $y = 0^j$, $z = 0^p 1^k$ with $i + j + p = k$.
  - By the pumping lemma, $xy^0 z \in L$ but $xy^0 z = 0^i 0^p 1^k$ and $i + p \neq k$ as $j = |y| \geq 1$, contradiction.
- $L = \{w \in \{0,1\}^* : |w| \text{ is a prime}\}$ is **not** regular.
- $L = \{ww^R : w \in \{0,1\}^*\}$ is **not** regular. $[w^R = w \text{ reversed}]$.

Some More Properties

Additional Properties of Regular Languages

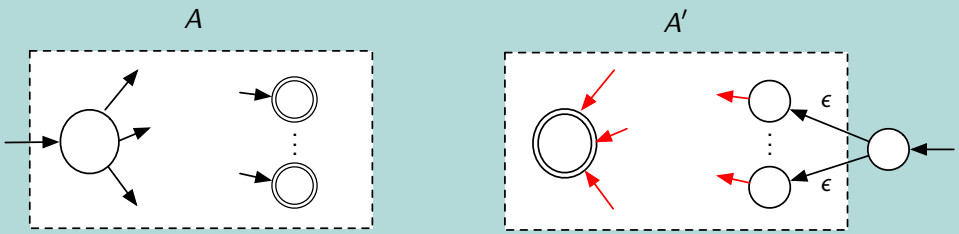
- We already know regular languages are closed under:
  - union, intersection, concatenation, Kleene-$*$ closure, and difference.
- We'll see three more operations under which regular languages are closed.
- Let L^R be the language obtained by reversing each string ($(01)^R = 10$)

Theorem 4.2.1

Let L be regular. Then $L^R := \{w^R : w \in L\}$ is also regular.

Proof of Theorem 4.2.1

- Let language L be accepted by DFA A.



- Let A' be the DFA obtained by: (a) Reversing each arrow in A; (b) swapping final and initial states; and (c) introduce ϵ-transitions to make initial state (of A') unique.
- Then L^R is accepted by A'. (They a small example yourself!)

Closure under Homomorphisms

› A homomorphism is a map $h : \Sigma_1 \to \Sigma_2^*$ that also satisfies that h can be extended to strings by defining $s_1 \cdots s_k \overset{h}{\mapsto} h(s_1) \cdots h(s_k)$.

Theorem 4.2.2

Let L be regular. Then $h(L) := \{h(w) : w \in L\}$ is also regular.

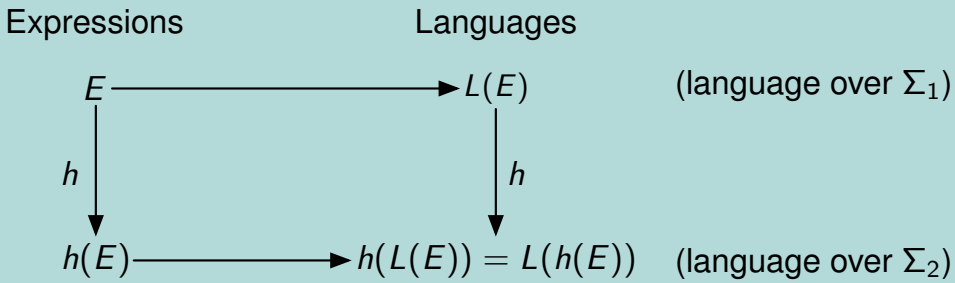
Proof of Theorem 4.2.2

› Let E be a regular expression corresponding to L.

› Let $h(E)$ be the expression obtained by replacing symbols $s \in \Sigma_1$ by $h(s)$.

› Then $h(E)$ is a regular expression over Σ_2. Thus, $L(h(E))$ is regular.

› It remains (but is easy) to show that $L(h(E)) = h(L(E))$



Closure under Inverse Homomorphisms

Theorem 4.2.3

Let L be regular. Then $h^{-1}(L) := \{w : h(w) \in L\}$ is also regular.

Proof of Theorem 4.2.3

- Let DFA $A = (Q, \Sigma_2, \delta, q_0, F)$ accept L
- Let DFA $B = (Q, \Sigma_1, \gamma, q_0, F)$ where

$$\gamma(q, s) = \hat{\delta}(q, h(s))$$

[Depending on the input, B mimics none, one, or many transitions of A]

- By definition, $\epsilon \in L(A)$ iff $q_0 \in F$ iff $\epsilon \in L(B)$
- By induction, we can show that

$$s_1 \cdots s_k \in L(B) \Leftrightarrow h(s_1) \cdots h(s_k) \in L(A) = L$$

- Hence, B accepts $h^{-1}(L)$.

Decision Properties of Regular Languages

Decision Properties

- DFAs and regular expressions are **finite representations** of regular languages
- How do we ascertain if a particular property is satisfied by a language?
  - Is the language accepted by a DFA non-empty?
  - Does the language accepted by a DFA contain a given string w?
  - Is the language accepted by a DFA infinite?
  - Do two given DFAs accept the same language?
  - Given two DFAs A and B, is $L(A) \subseteq L(B)$?
- We will look at the above 5 questions assuming that regular languages are defined by DFAs. (If the language is specified by an expression, we can convert it to a DFA!)

Decision Properties (Emptiness, Membership, Infiniteness)

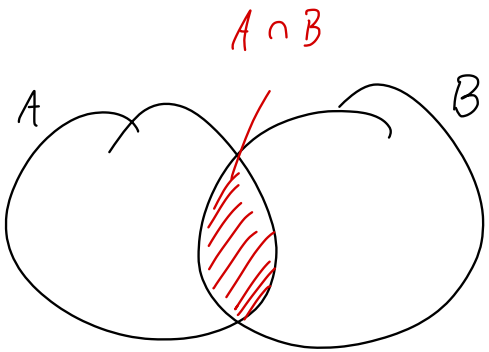
- **Emptiness:** Identify all reachable states, possible in $O(n^2)$ time (with n states). L is empty if none is final.
- **Membership:** Simply execute w and check resulting state. Possible in $O(|w|)$ time.
- **Infiniteness:** We can reduce the problem of infiniteness to finding a cycle in the directed graph (a.k.a. transition diagram) of the DFA.
  - First, delete any node unreachable from the initial node ($O(n^2)$ complexity).
  - Next, delete nodes that cannot reach any final node ($O(n^3)$ complexity).
  - Use depth-first search (DFS) to find a cycle in the remaining graph ($O(n^2)$ complexity).
- **Q:** How do runtimes change if we have an NFA? Should we compile? (Homework!)

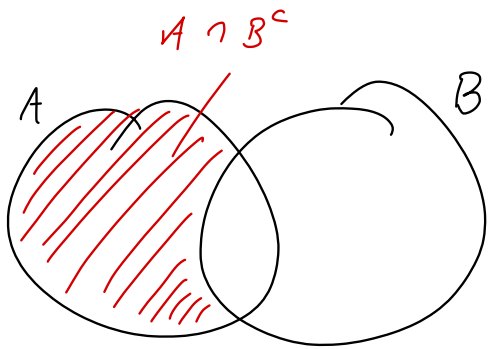
Decision Properties (Equivalence)

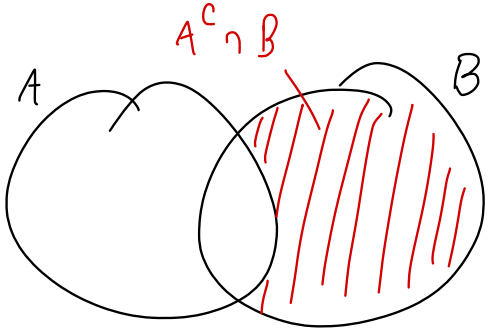
› **Equivalence:** Given $A = (Q_A, \Sigma, \delta_A, q_{A0}, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_{B0}, F_B)$, how do we ascertain if $L(A) = L(B)$?

$$L(A) = L(B) \Leftrightarrow \begin{array}{l} L(A) \cap L(B)^c = \emptyset \\ L(A)^c \cap L(B) = \emptyset \end{array}$$

Why is this true?







Thus:

$A \cap B = A = B$ #

$A \cap B^c = A^c \cap B = \emptyset$

Decision Properties (Equivalence)

- **Equivalence:** Given $A = (Q_A, \Sigma, \delta_A, q_{A0}, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_{B0}, F_B)$, how do we ascertain if $L(A) = L(B)$?

$$L(A) = L(B) \Leftrightarrow \begin{array}{l} L(A) \cap L(B)^c = \emptyset \\ L(A)^c \cap L(B) = \emptyset \end{array}$$

Run A and B in parallel. (Not their complement-versions!)

  - $L(A) \cap L(B)^c$: Accept if resp. paths end in F_A and F_B^c.
  - $L(A)^c \cap L(B)$: Accept if resp. paths end in F_A^c and F_B.

- Use **product** DFA: Construct $C = (Q_C, \Sigma, \delta_C, q_{C0}, F_C)$ defined by

$$\begin{aligned} Q_C &= Q_A \times Q_B && \text{[Cartesian Product]} \\ q_{C0} &= (q_{A0}, q_{B0}) \\ \delta_C((q, q'), s) &= (\delta_A(q, s), \delta_B(q', s)) && \text{[Both DFAs are simulated in parallel]} \\ F_C &= (F_A \times F_B^c) \cup (F_A^c \times F_B) && \text{[accept strings in exactly one of } L(A) \text{ or } L(B)\text{]} \end{aligned}$$

$$L(A) = L(B) \Leftrightarrow L(C) = \emptyset$$

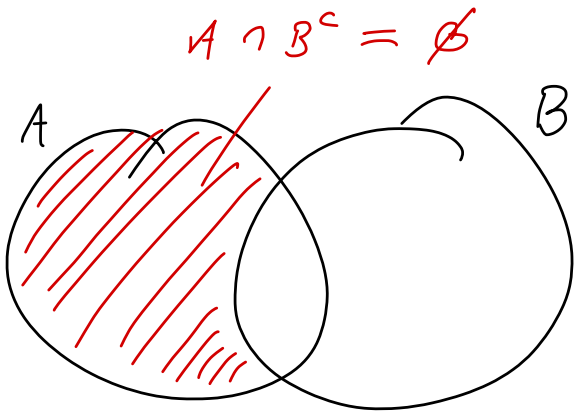
(and you know how to check for an empty language of a DFA!)

Decision Properties (Inclusion)

› **Inclusion:** Given $A = (Q_A, \Sigma, \delta_A, q_{A0}, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_{B0}, F_B)$, how do we ascertain if $L(A) \subseteq L(B)$?

$$L(A) \subseteq L(B) \;\Leftrightarrow\; L(A) \cap L(B)^c = \emptyset$$

Why is this true?



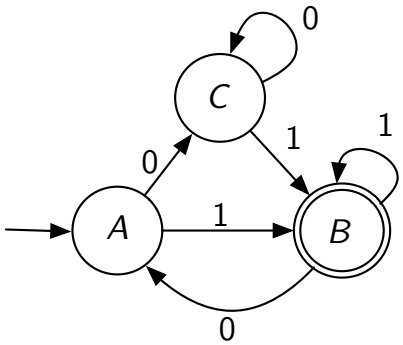
Thus: $A \subseteq B$

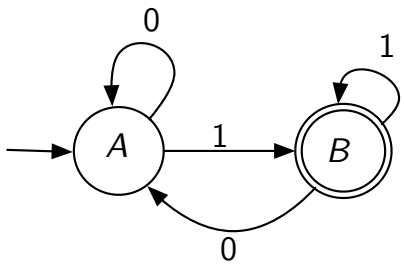
What now? Everything as before! Only now $F_C = F_A \times F_B^c$.

Minimizing the Number of States

DFA State Minmimization

- Given two DFAs, we know how to test if they accept the same language.
- Is there a unique minimal DFA for a given regular language?
- Given a DFA, can we **reduce** the number of states without altering the language it accepts?





The two DFAs accept the same language and hence state C is unnecessary.

- How do we (identify and) remove 'unnecessary' states without altering the underlying language?

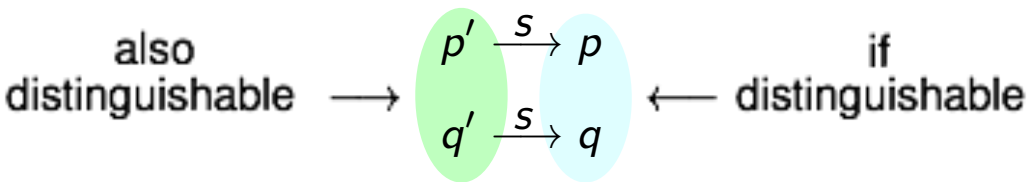
DFA State Minimimization

- State minimization requires a notion of **equivalence** or **distinguishability** of states.
- We define **distinguishability** based on their words' **finality**

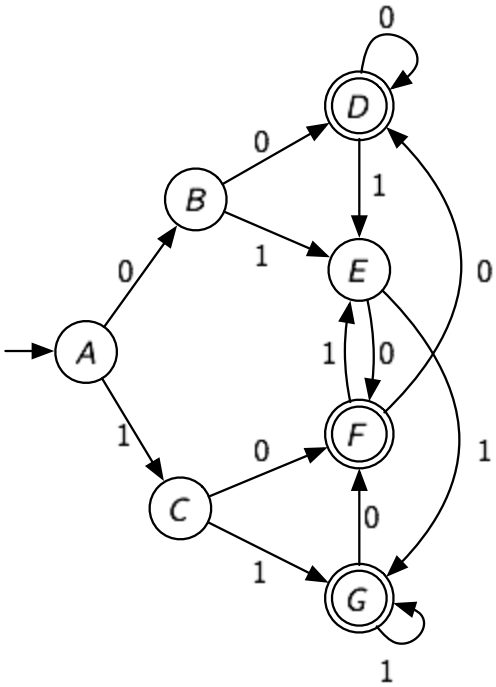
states p and q are **equivalent** or indistinguishable $\Leftrightarrow$ $\hat{\delta}(p, w) \in F$ whenever $\hat{\delta}(q, w) \in F$ for every $w \in \Sigma^*$.

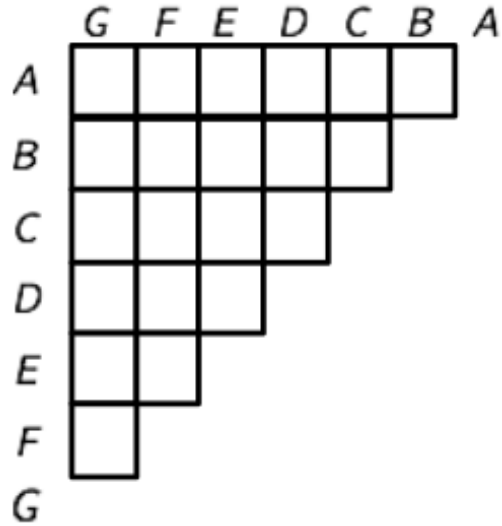
states p and q are **distinguishable** $\Leftrightarrow$ exactly one of $\hat{\delta}(p, w)$ or $\hat{\delta}(q, w)$ is in F for some $w \in \Sigma^*$.

- **Table Filling** Algorithm identifies equivalent and distinguishable pairs of states.
  - Any final state is **distinguishable** from a non-final state (and vice versa)
  - If (a) p and q are distinguishable; and there exist states p', q', and symbol s such that (b) $\hat{\delta}(p', s) = p$ and (c) $\hat{\delta}(q', s) = q$, then p' and q' are also distinguishable



Identifying pairs of (In)distinguishable States: An Example

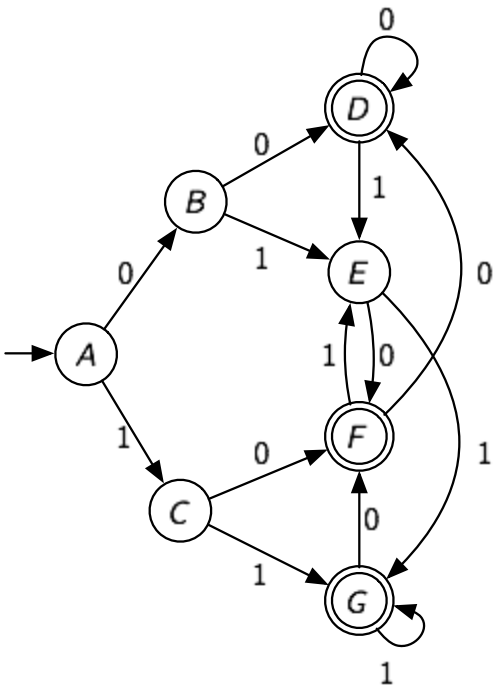




- Fill in $\times$ whenever one component of pair is final, and other is not.
- Fill in $\times$ if 1 moves the pair of states to a distinguishable pair
- Fill in $\times$ if 0 moves the pair of states to a distinguishable pair
- Repeat until no progress (more precise "filling order": see next slide)

(This slide is added to the handout so you can try it yourself!)

Identifying pairs of (In)distinguishable States: An Example



	G	F	E	D	C	B	A
A	×	×	×	×	×	×	
B	×	×	×	×	×		
C	×	×		×			
D	×		×				
E	×	×					
F	×						
G							

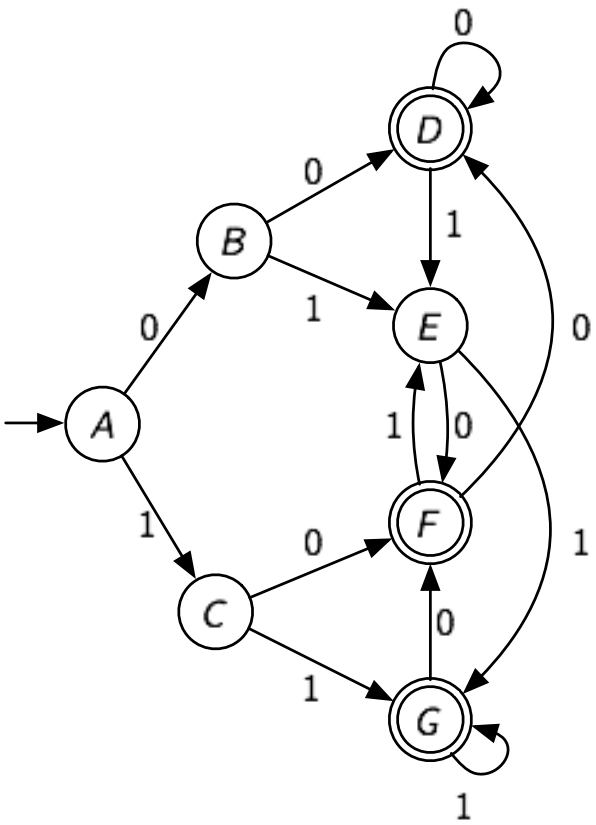
- Fill in × whenever one component of pair is final, and other is not.
- Fill in × if 1 moves the pair of states to a distinguishable pair
- Fill in × if 0 moves the pair of states to a distinguishable pair
- Repeat until no progress (more precise "filling order": see next slide)

Theorem 4.4.1

Any two states without a × sign are equivalent.

- Proof idea: If two states are distinguishable, the algorithm **will** fill a × eventually.

Table-filling Algorithm



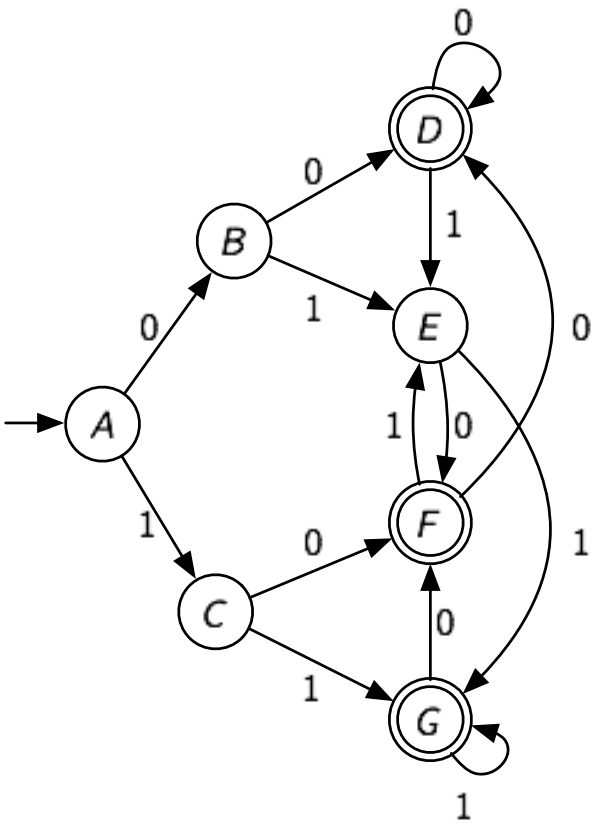
- Delete states not reachable from start states
- Delete any non-starting state that cannot reach any final state
- Find distinguishable and equivalent pairs of states

Do this in a systematic way, for the sake of uniqueness (the set of cells remaining empty will remain the same, but different orders will result into differently colored patterns), which helps us mark. Different orders may however result in differently many iterations!

We always start in row 1 and proceed column by column, then do row 2, etc.

The symbol we start with is given by us.

Table-filling Algorithm

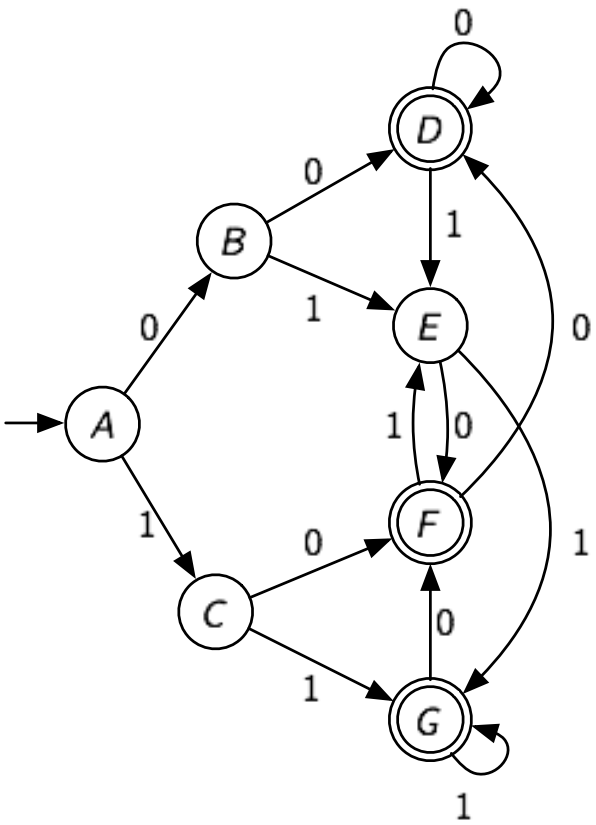


- Delete states not reachable from start states
- Delete any non-starting state that cannot reach any final state
- Find distinguishable and equivalent pairs of states
- Find equivalence classes of indistinguishable states. In this example: $\{A\}, \{B\}, \{C, E\}, \{D, F\}, \{G\}$

	G	F	E	D	C	B	A
A	×	×	×	×	×	×	
B	×	×	×	×	×		
C	×	×		×			
D	×		×				
E	×	×					
F	×						
G							

Color-blind?
red: A/B, blue: A/E, A/C, B/E, B/C, D/G, F/G

Table-filling Algorithm



- Delete states not reachable from start states
- Delete any non-starting state that cannot reach any final state
- Find distinguishable and equivalent pairs of states
- Find equivalence classes of indistinguishable states. In this example: $\{A\}, \{B\}, \{C, E\}, \{D, F\}, \{G\}$
- Collapse each equivalence class of states to a state
- Delete parallel transitions with same label.

Remark: The resultant transition diagram will be a DFA.

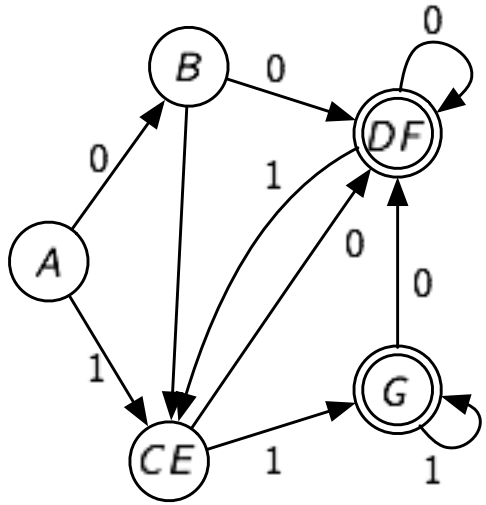


Table-filling: Other Uses

- Test equivalence of languages accepted by 2 DFAs.
  - Given $A = (Q_A, \Sigma, \delta_A, q_{A0}, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_{B0}, F_B)$:
    - Rename states in Q_B so that Q_A and Q_B are disjoint.
    - View A and B together as one DFA
      (Ignore the fact that there are 2 start states)
    - Run table-filling on $Q_A \cup Q_B$.
    - q_{A0} and q_{B0} are indistinguishable $\Leftrightarrow L(A) = L(B)$.

  [Why?] If w distinguishes q_{A0} from q_{B0} then w cannot be in both $L(A)$ and $L(B)$
- Suppose a DFA A cannot be minimized further by table-filling. Then, A has the least number of states among all DFAs that accept $L(A)$