COMP3630 / COMP6363

week 2: **Context-free Grammars and Languages**

This Lecture Covers Chapter 5 of HMU: Context-free Grammars and Languages

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The Australian National University

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Content of this Chapter

- (Context-free) Grammars
- (Leftmost and Rightmost) Derivations
- Parse Trees
- An Equivalence between Derivations and Parse Trees
- Ambiguity in Grammars

Additional Reading: Chapter 5 of HMU.

Grammars

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- **Grammars** are a generative means of defining languages.
- Grammars can be used to create a strictly larger class of languages.
- They are especially useful in compiler and parser design; they can be used to check if:
  - parentheses are balanced in a program,
  - `else` occurrences have a matching `if`, etc.

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Notation

- *Strings consisting of non-terminals and/or terminals will be denoted by greek symbols, e.g.,* $\alpha, \beta, \ldots$.
- *Strings of terminals will be denoted by lower case letters, e.g.,* w, u, v

Derivations

How do Grammars Generate Languages?

- A string $w \in T^*$ is in the language $L(G)$ generated by $G = (V, T, \mathcal{P}, S)$ iff we can **derive** w from S, i.e.,

  start from S and use production rule(s) repeatedly to replace heads of the rules by their bodies until a string in T^* is obtained.

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(1) $\left\{ \begin{array}{c} (S, \epsilon), (S, 0), (S, 1) \\ (S, 0S0), (S, 1S1) \end{array} \right\}$

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$$\begin{array}{l} S \longrightarrow \epsilon \\ S \longrightarrow 0 \\ S \longrightarrow 1 \\ S \longrightarrow 0S0 \\ S \longrightarrow 1S1 \end{array}$$

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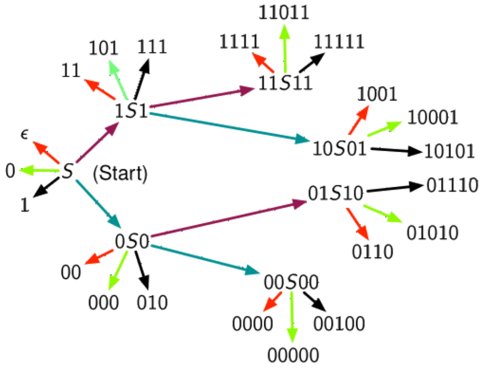

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In the above, $\uparrow$ indicates the variable that is replaced in the following step

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- *if an interior node is labelled by* $A \in V$*, and its children are labelled* $s_1, \ldots, s_k \in V \cup T \cup \{\epsilon\}$*, then* $A \longrightarrow s_1 \cdots s_k$ *is a production rule in* $\mathcal{P}$*.*

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The **yield** *of a parse tree is the string formed from the labels of the tree leaves read from left to right.*
Note: *The yield is not necessarily a string of terminals.*

$$G = (\{S\}, \{(,)\}, \mathcal{P}, S)$$
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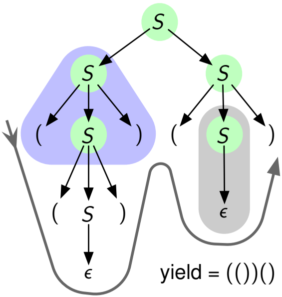


An Equivalence between Parse Trees and Derivations

Derivations and Parse Trees

- Parse trees, derivations, leftmost derivations, and rightmost derivations are equivalent means of generating words of the language $L(G)$ of a CFG G.
- The proof for equivalence of rightmost derivations mirrors that of leftmost derivations. (So we'll not delve into rightmost derivations).

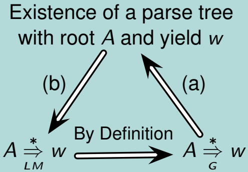
Theorem 5.5.1

Let CFG $G = (V, T, \mathcal{P}, S)$ be given. Let $A \in V$ and $w \in T^*$. Then,

$A \overset{*}{\underset{G}{\Rightarrow}} w \Leftrightarrow A \overset{*}{\underset{LM}{\Rightarrow}} w \Leftrightarrow$ *there exists a parse tree with root A and yield w* $\Leftrightarrow A \overset{*}{\underset{RM}{\Rightarrow}} w.$

Proof Idea

We'll show the following implications.



Part (a) of Proof of Theorem 5.5.1: $A \overset{*}{\underset{G}{\Rightarrow}} w \Rightarrow \exists$ Parse Tree

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Lemma 5.5.2

Let CFG $G = (V, T, \mathcal{P}, S)$ be given. Let $A \in V$ and $\alpha \in SF(G)$. If $A \overset{*}{\underset{G}{\Rightarrow}} \alpha$, then there exists a parse tree with root A and yield α.

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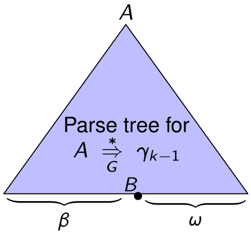

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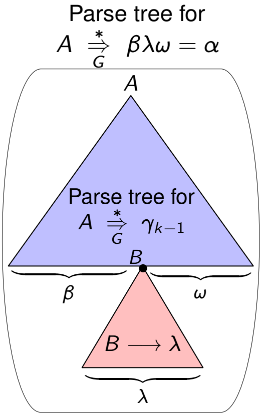
- Extend the parse tree from the first $k-1$ steps by:
  - If $\lambda = X_1 \ldots X_n$ with $X_1, \ldots, X_n \in V \cup T$, add childen $X_1, \ldots, X_n$ to node B.



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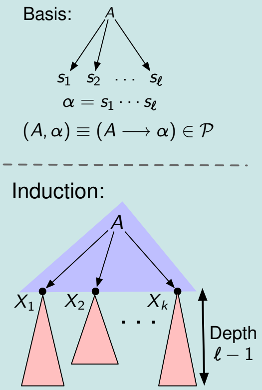

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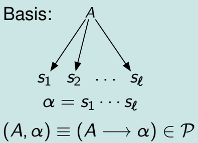

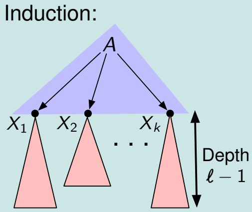

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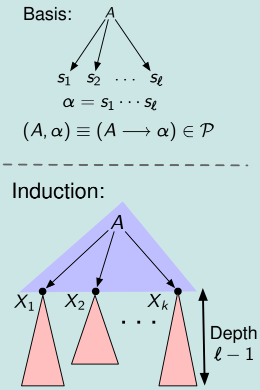

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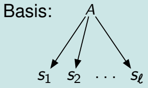

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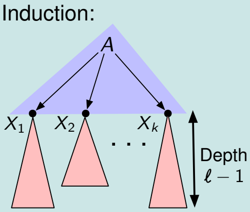

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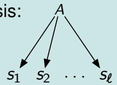
Basis:



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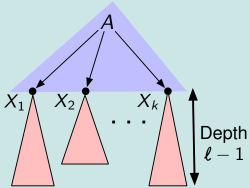
Induction:



Then, the following is a leftmost derivation for α from A

$$A \underset{G}{\Rightarrow} X_1 X_2 \cdots X_k \overset{*}{\underset{LM}{\Rightarrow}} w_1 X_2 \cdots X_k \overset{*}{\underset{LM}{\Rightarrow}} w_1 w_2 X_3 \cdots X_k \overset{*}{\underset{LM}{\Rightarrow}} \cdots \overset{*}{\underset{LM}{\Rightarrow}} w_1 \cdots w_k$$

Ambiguous Grammars

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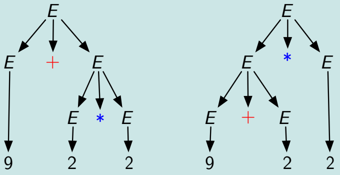

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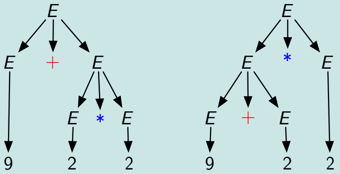

- This ambiguity is addressed by precedence rules for operators.

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$$B \overset{*}{\underset{LM}{\Rightarrow}} (RB \underset{LM}{\Rightarrow} ()B \underset{LM}{\Rightarrow} ()(RB \underset{LM}{\Rightarrow} ()()B \underset{LM}{\Rightarrow} ()()()B \underset{LM}{\Rightarrow} ()()()\epsilon$$

- Some languages are intrinsically ambiguous, e.g., $\{0^i1^j2^k : i = j \text{ or } j = k\}$. All grammars for such languages are ambiguous.
- In general, there is **no** way to tell if a grammar is ambiguous.