COMP3630 / COMP6363

week 3: **Pushdown Automata**

This Lecture Covers Chapter 6 of HMU: Pushdown Automata

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The Australian National University

Semester 1, 2025

Content of this Chapter

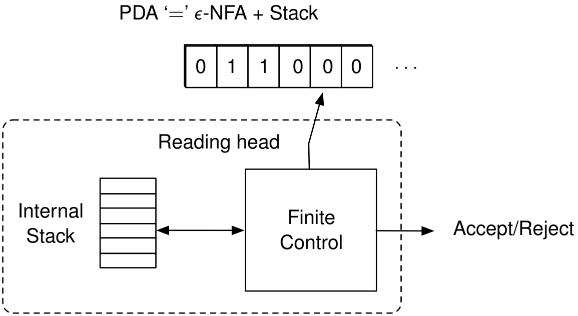
- Pushdown Automata (PDA)
- Language accepted by a PDA
- Equivalence of CFGs and the languages accepted by PDAs
- Deterministic PDAs

Additional Reading: Chapter 6 of HMU.

Pushdown Automata

Introduction to PDAs

- PDA '=' ϵ-NFA + Stack (LIFO)



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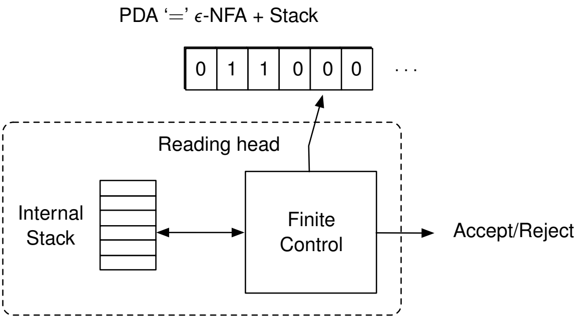

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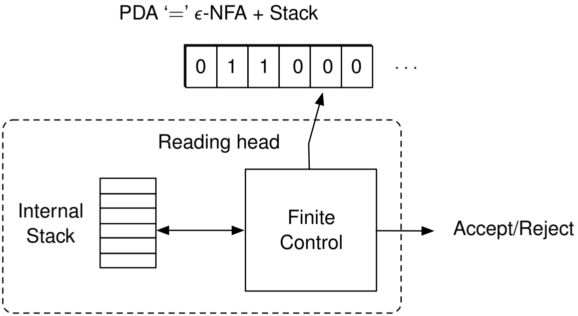

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- Note: The PDA can only read a symbol once (i.e., the reading head is unidirectional).

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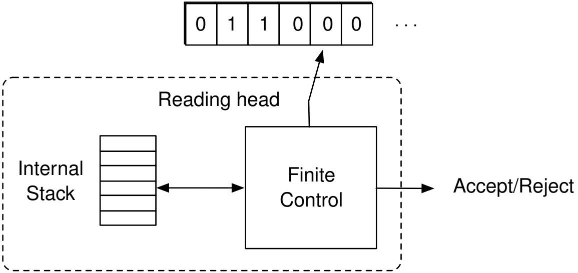

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- Γ is the finite alphabet of stack symbols;
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$ is a partial function such that $\delta(q, a, \gamma)$ (if defined) is a finite set of pairs $(q', \gamma') \in Q \times \Gamma^*$. **// This is non-deterministic! Why?**
- We have a partial function since we don't need to define transitions for all possible values in $Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma$.
- However, note that using a "normal" (i.e., non-partial) function is still correct when defining $\delta(q, a, A) = \emptyset$ for $q \in Q$, $a \in \Sigma^* \cup \{\epsilon\}$, and $A \in \Gamma$ whenever we want no transition for $\delta(q, a, A)$. (Since we don't actually have a transition in these cases when mapping to an empty set.)

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- $Z_0 \in \Gamma$ is the sole symbol atop the stack at the start; and

$$\delta(q, a, A) = \{(q'_i, \gamma_i) : i = 1, \ldots, \ell\}$$

Present state: q

Input symbol (or ϵ): a

Stack symbol on top: A

Next state: q'_i

The string replacing A on top of the stack: γ_i

The number of possible transitions: ℓ

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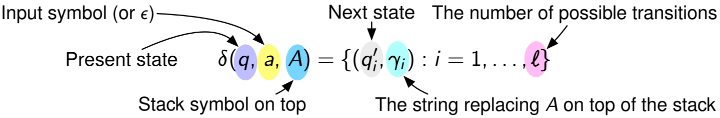


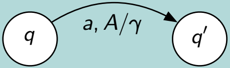
Convention: lower case symbols s, a, and b will denote input symbols; lower case symbols u, v, w, x, ... will exclusively denote strings (sequences!) of input symbols; stack symbols are indicated by upper case letters (e.g., A, B, etc); strings of stack symbols are indicated by greek letters (e.g., α, β, etc);

A PDA Example

Transition Diagram Notation

Notation: The label $a, A/\gamma$ on the edge from a state q to q' indicates a possible transition from state q to state q' by reading the symbol a when the top of the stack contains the symbol A. This stack symbol is then replaced by the string γ.

$$(q', \gamma) \in \delta(q, a, A) \quad \Leftrightarrow$$



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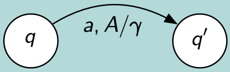

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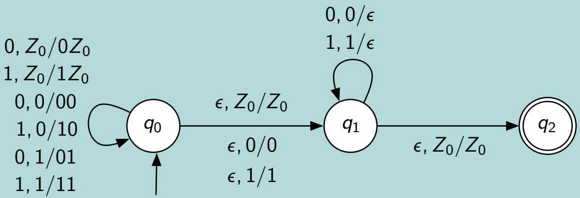
PDA that accepts ???



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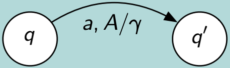

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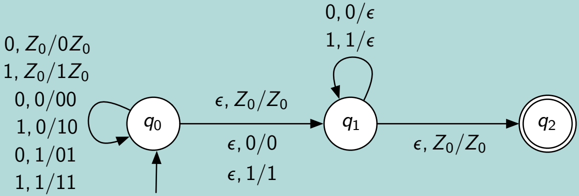
PDA that accepts $L = \{ww^R : w \in \{0.1\}^*\}$



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- **One-step computation** of a PDA P, denoted by $\underset{P}{\vdash}$, indicates configuration change due to one transition. Suppose $(q', \gamma) \in \delta(q, a, A)$. For $w \in \Sigma^*$, $\alpha \in \Gamma^*$,
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- **(multi-step) computation**, denoted by $\overset{*}{\underset{P}{\vdash}}$, indicates configuration change due to zero or any finite number of consecutive PDA transitions.
  - $ID \overset{*}{\underset{P}{\vdash}} ID'$ if there are k IDs $ID_1, \ldots, ID_k$ (for some $k \geq 1$) such that:
    (i) $ID_1 = ID$ and $ID_k = ID'$, and
    (ii) for each $i = 1, \ldots, k-1$, $ID_i \underset{P}{\vdash} ID_{i+1}$.

Beware of IDs!

Lemma 6.2.1

Let PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *be given. Let* $q, q' \in Q$, $x, y, w \in \Sigma^*$, *and* $\alpha, \beta, \gamma \in \Sigma^*$. *Then the following hold.*

$$(q, x, \alpha) \overset{*}{\underset{P}{\vdash}} (q', y, \beta) \quad \Leftrightarrow \quad (q, xw, \alpha) \overset{*}{\underset{P}{\vdash}} (q', yw, \beta) \tag{1}$$

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- (1) What hasn't been read cannot affect configuration changes
- (2) PDA transitions cannot occur on empty stack. So the $(q, x, \alpha) \overset{*}{\underset{P}{\vdash}} (q', y, \beta)$ must not access any location beneath the last symbol of x.

Remark: Think about why is the reverse implication of (2) not true.

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Given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, the language accepted by P <u>*by final states*</u> *is*

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- Which of the two definitions accepts 'more' languages?

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Assuming accepting by final state would be exactly as powerful as accepting by empty stack, what would have to hold?

Let P be a PDA.

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Taken both together we have, if true, that both criteria are equally expressive.

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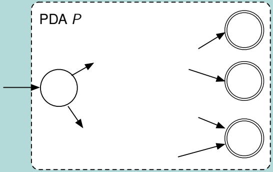

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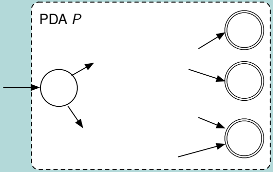

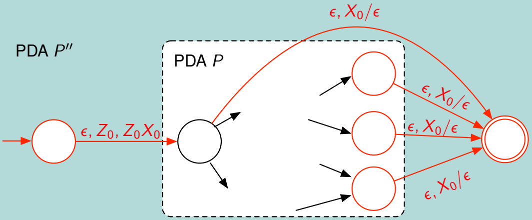

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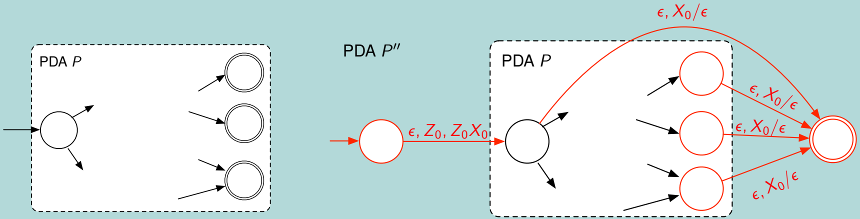
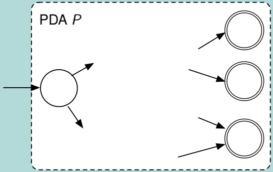


- Introduce a new start state and a new final state with the transitions as indicated.
- The start state first replaces the stack symbol Z_0 by Z_0X_0.
- If and only if $w \in N(P)$ will the computation by P end with the stack containing precisely X_0.
- The PDA P'' then transitions to the final state popping X_0. Hence, $N(P) = L(P'')$.

Equivalence of the two Notions of Language Acceptance

Proof of Existence of P' such that $L(P) = N(P')$ (i.e., P accepts by final states)



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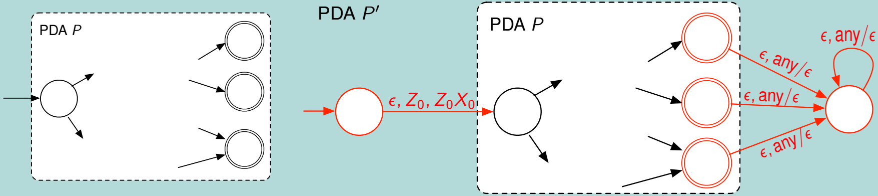

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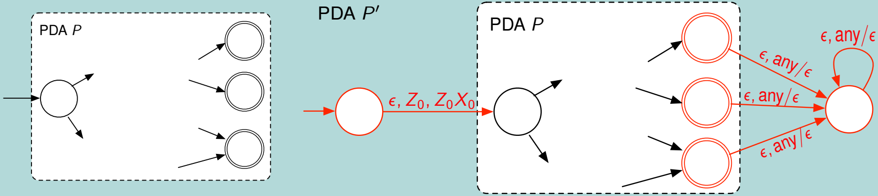

- Introduce a new start state and a special state with the transitions as indicated.
- The start state first replaces the stack symbol Z_0 by Z_0X_0.
- If and only if $w \in L(P)$ will the computation by P end in a final state with the stack containing (at least) X_0. **Question:** Why is this required?
- The PDA P' then transitions to the special state and starts to pop stack symbols one at time until the stack is empty. Hence, $L(P) = N(P')$.

CFGs and PDAs

CFGs and PDAs

Is every CFL accepted by some PDA and vice versa?

Theorem 6.3.1

For every CFG G, there exists a PDA P such that $N(P) = L(G)$.

Proof

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$$[\text{Type 1}]\ \ \delta(q_0, a, a) = \{(q_0, \epsilon)\}, \quad \text{whenever } a \in \Sigma,$$

$$[\text{Type 2}]\ \ \delta(q_0, \epsilon, A) = \{(q_0, \alpha) : A \longrightarrow \alpha \text{ is a production rule in } \mathcal{P}\}.$$

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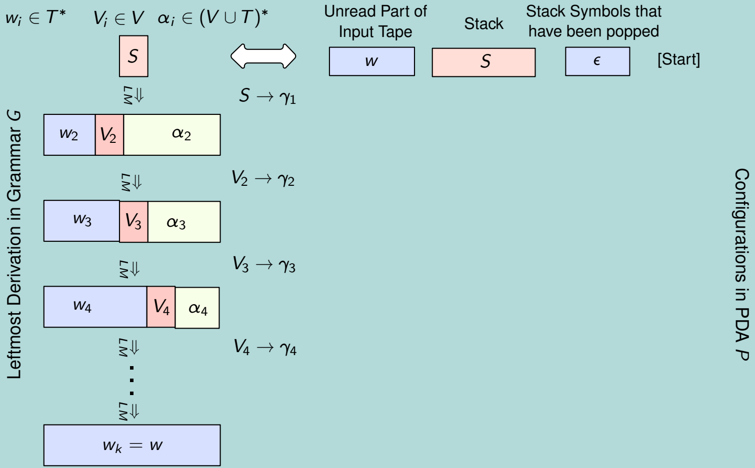
Remark: That's a great (fun!) exercise to practice by yourself! Just take any grammar.

CFGs and PDAs (see appendix slide!)

Proof of 1-1 Correspondence between PDA Moves and Leftmost Derivations

Suppose $w \in T^*$ and $S \overset{*}{\underset{LM}{\Rightarrow}} w$. $x \setminus y$:=suffix of y in x.



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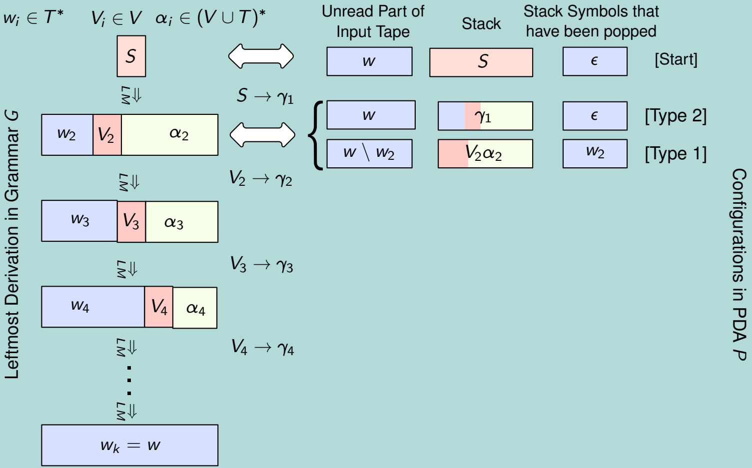

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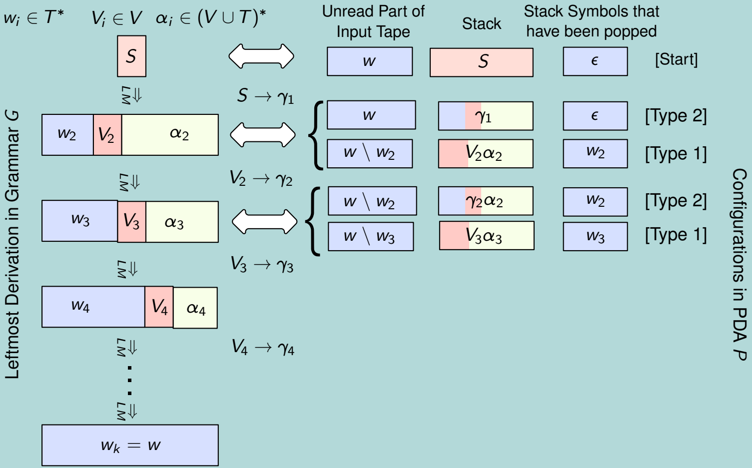

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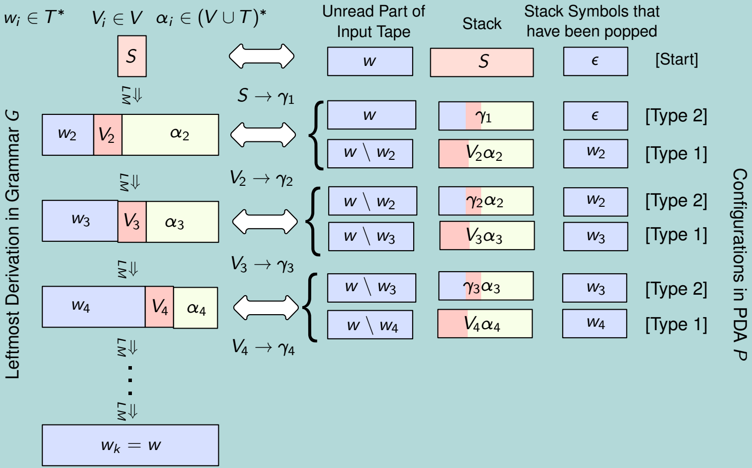

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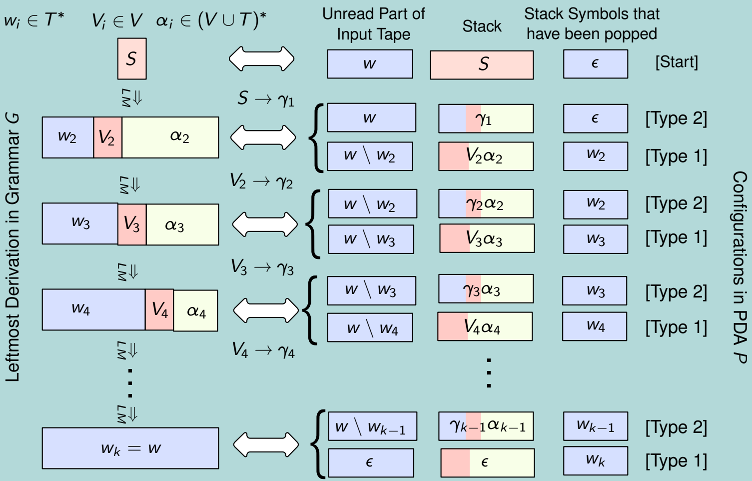

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- Given $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, we define $G = (V, T, \mathcal{P}, S)$ as follows.
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      Note that if $(r, \epsilon) \in \delta(q, a, X)$, then $[qXr] \longrightarrow a$.

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- **Remark:** Not 100% clear? Translate a small PDA! (One with few states.)

CFGs and PDAs (see appendix slide!)

Proof of $(q, w, X) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon) \Rightarrow [qXp] \overset{*}{\underset{G}{\Rightarrow}} w$. (Induction on # of steps of computation)

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2. A portion of x is read, and Y_1 is popped; more is read, Y_2 is popped,...

$$(r_1, \overbrace{w_1 w_2 \cdots w_\ell}^{=x}, Y_1 Y_2 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_2, w_2 \cdots w_k, Y_2 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_3, w_3 \cdots w_k, Y_3 \cdots Y_k) \cdots \overset{*}{\underset{P}{\vdash}} (r_k, w_k, Y_k) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon)$$

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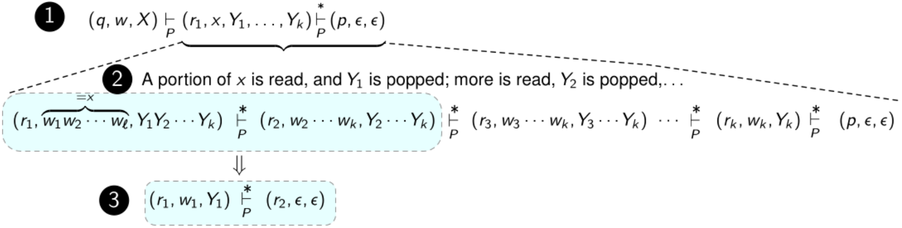

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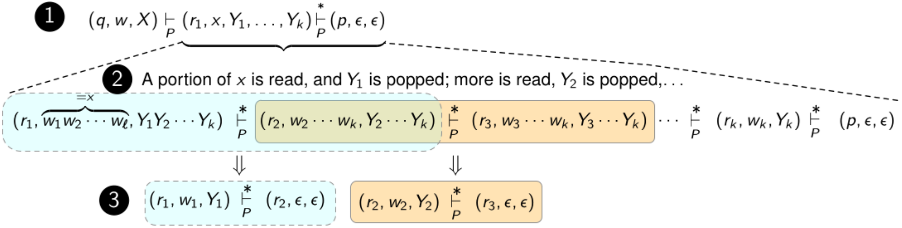

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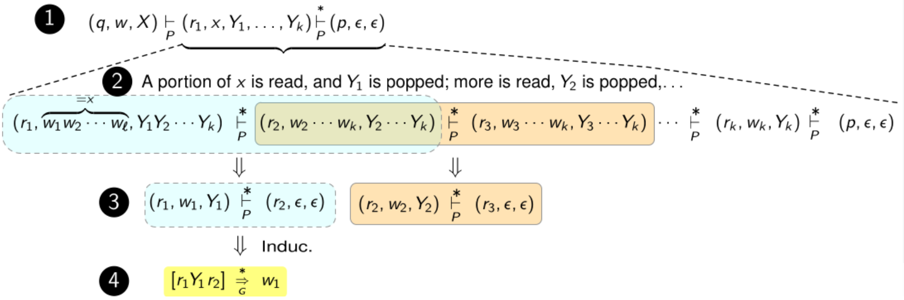

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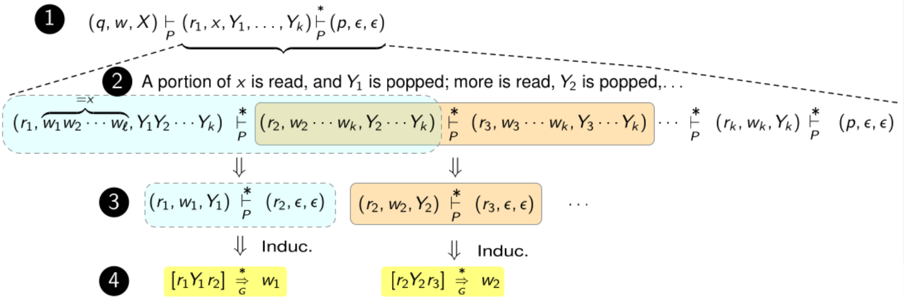

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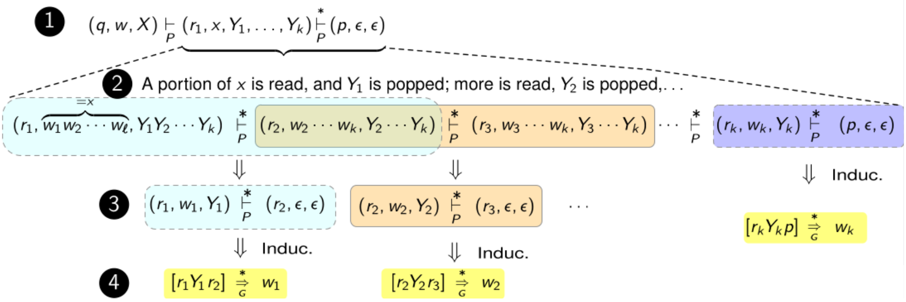

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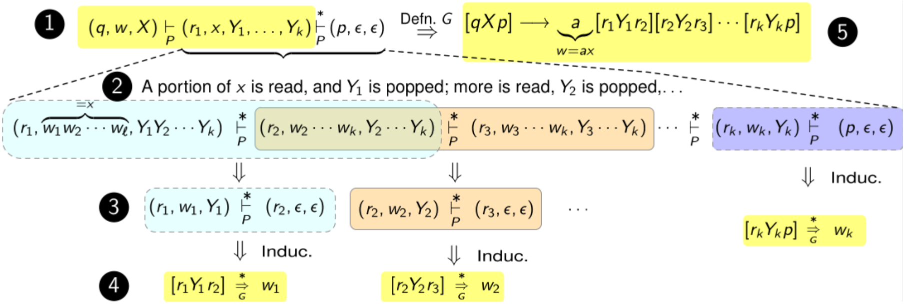

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- Induction: Let $[qXp] \overset{*}{\underset{G}{\Rightarrow}} w$.

❶ $[qXp] \underset{LM}{\Rightarrow} a\,[r_0Y_1r_1]\,[r_1Y_2r_2] \cdots [r_{k-1}Y_kp] \overset{*}{\underset{LM}{\Rightarrow}} w$

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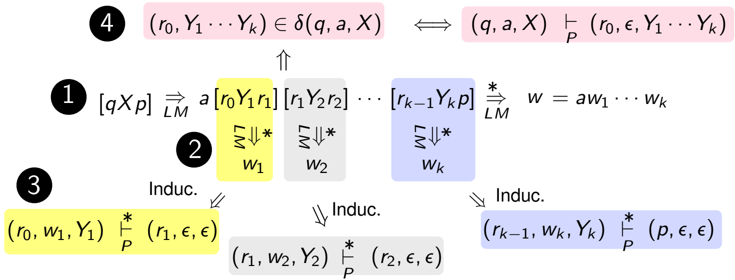

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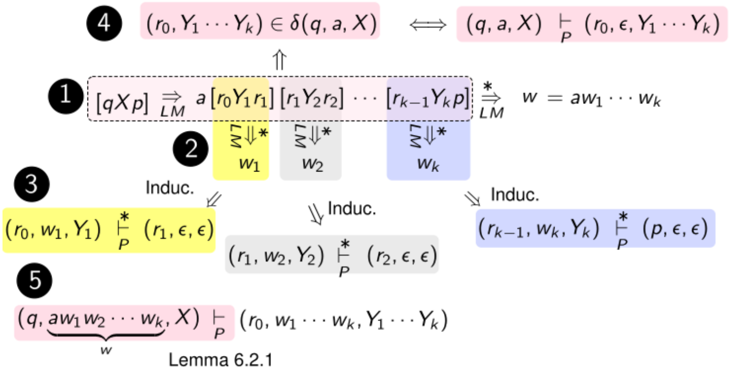

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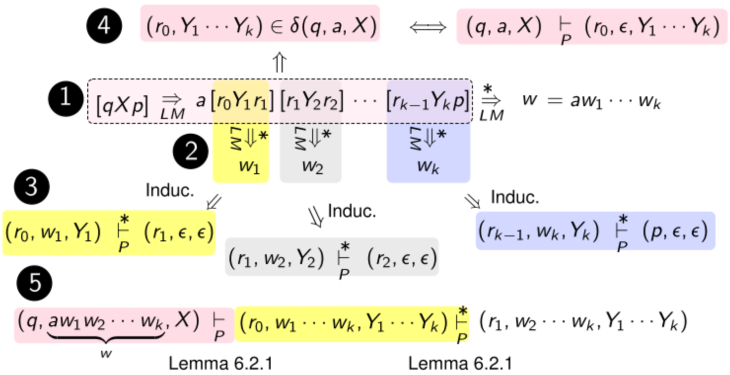

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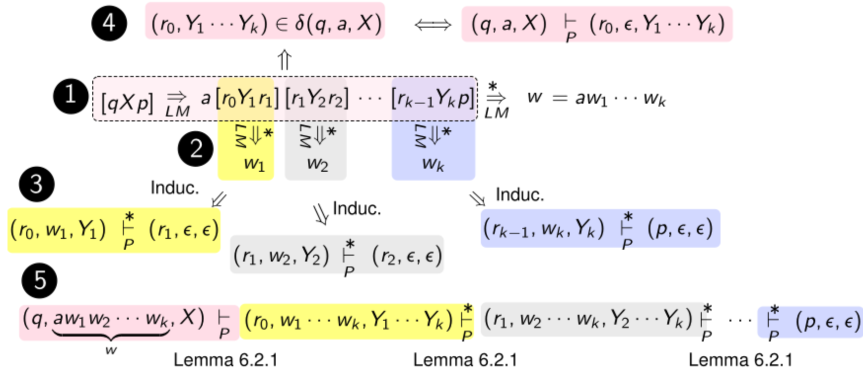

Deterministic PDAs

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Definition

A DPDA P is a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ such that for each $q \in Q$ and $X \in \Gamma$,

- $|\delta(q, a, X)| \leq 1$ *for any* $a \in \Sigma \cup \{\epsilon\}$,
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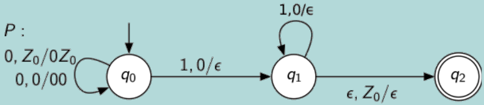

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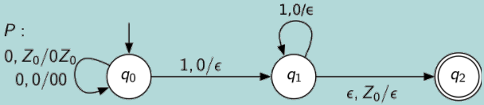

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Languages Accepted by DPDAs

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Theorem 6.4.1

Every regular language L is the language accepted by some DPDA accepting by final states.

Proof

Simply view the DFA accepting L as a DPDA (with the stack always containing Z_0).

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Proof

- Let $L = \{0\}^*$ (which is regular). It cannot equal $N(P)$ for any DPDA P.
- Suppose DPDA P accepts L by emptying its stack. Since 0 is accepted, P eventually reaches a configuration (p, ϵ, ϵ) for some state p.
- Now, suppose that P is fed with the input 00. Since P is **deterministic**, P reads a 0 and eventually has to get to $(p, 0, \epsilon)$. However, it hangs at this configuration and cannot read any further input symbols. Hence, P cannot accept 00.

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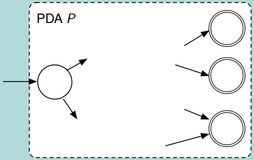

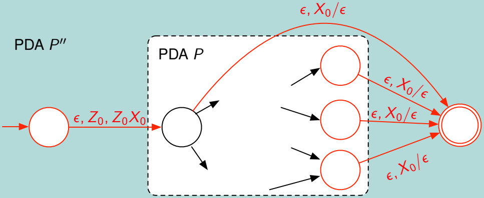

Languages Accepted by DPDAs

Proof $\Leftarrow$

$\Leftarrow$ Let DPDA P'' be given. Let $w \in L(P'')$, $(q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \gamma)$ for some $p \in F$, and $\gamma \in \Gamma$.

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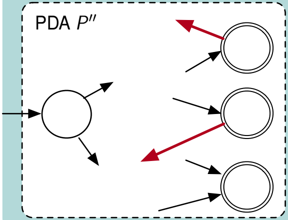
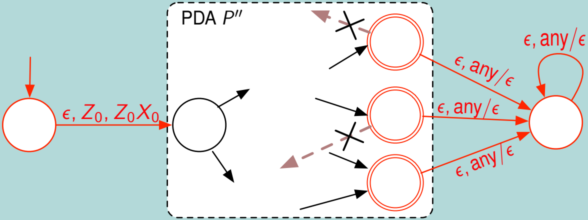
- Then we can delete all transitions from final states; this does not alter $L(P'')$.
- Then, the construction of Theorem 6.2.2 yields a **deterministic** PDA P' such that $N(P') = L(P'') = L$.





DPDAs and Unambiguous Grammars

Theorem 6.4.4

If $L = N(P)$ for some DPDA P, then L has an unambiguous CFG.

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- Suppose G is ambiguous. Then, some $w \in L$ has 2 leftmost derivations.
- However, each derivation corresponds to a unique trajectory of configurations in P that also accepts w by emptying the stack.
- Since P is deterministic, the trajectories, and hence, the derivations have to be identical. Hence, G is unambiguous.

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DPDAs and unambiguous Grammars

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If $L = L(P)$ for some DPDA P, then L has an unambiguous CFG.

Proof

- Let $\$$ be a symbol not in the alphabet of L.
- Consider $L' = \{w\$: w \in L\}$. Then, L' has the prefix property.
- By Theorem 6.4.3, there must exist a DPDA P' such that $L' = N(P')$.
- By Theorem 6.4.4, L' has an unambiguous CFG $G' = (V, T, \mathcal{P}, S)$.
- Define CFG $G = (V \cup \{\$\}, T \setminus \{\$\}, \mathcal{P} \cup \{\$ \longrightarrow \epsilon\}, S)$. G generates L.
- Proof by contradiction: Suppose G is ambiguous.
- Then, some $w \in L$ has 2 leftmost derivations.
- The last steps in the two leftmost derivations of w **must** use the production $\$ \longrightarrow \epsilon$. (So this can't cause ambiguity.)

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- Thus, the portions of the two leftmost derivations without the last production step (which corresponds to G') must allow two different LM derivations.
- Hence, G' must be ambiguous, contradiction! Hence, G must unambiguous.

Additional Slides

Explanation for Slide 16

- $\Rightarrow$ Suppose we want to show that if there is a derivation in G generating w, then there is a trajectory in P accepting w. To do that let $S \overset{*}{\underset{LM}{\Rightarrow}} w$.
- Then there must be a LM derivation as in the left column. In each step of the leftmost derivation, a part of the string w is uncovered, and the uncovered part is succeeded by a non-terminal.
- Let after $i = 1, \ldots, k-2$ production uses: (1) the prefix w_{i+1} of w be uncovered (shown in purple); (2) the leftmost non-terminal be V_{i+1} (shown in orange); and (3) is the string to the right of the leftmost non-terminal α_{i+1} that contains both terminal and non-terminal symbols (shown in beige).
- After the k^{th} production rule, we have derived $w_k = w$.
- Now suppose $S \to \gamma_1 = w_2 V_2 \alpha_2$, $V_2 \to \gamma_2$, ..., $V_{k-1} \to \gamma_{k-1}$ be the $k-1$ production rules used in the leftmost derivation.
- Now let us show that a trajectory exists for P using the above information we have laid out.
- Since there is only one state for the PDA, the right part of the slide presents only the portion of tape yet to be read, and the stack contents; additionally, it also gives the **string of terminals** that has been popped up until any point in time.
- Initially, the tape contains w, the stack contains S, and ϵ has been popped thus far.

Explanation for Slide 16 (Continued)

- Now since $S \to \gamma_1$ is a valid production rule, by the definition of P, there is a Type-22 transition that reads nothing from the input tape, reads S from the stack and pushes $\gamma_1 := w_2 V_2 \alpha_2$ onto the stack. Thus, the following one-step computation is valid

$$(q_0, w, S) \underset{P}{\vdash} (q_0, w, w_2 V_2 \alpha_2).$$

- Note that w_1 is the prefix of w uncovered after the first step of the derivation, and hence matches the first few symbols of w. Then, it is clear that one can perform $|w|$ Type-1 transitions that pop each of these symbols from the stack. Thus, after popping $|w_1|$ symbols, we see that:

$$(q_0, w, S) \underset{P}{\vdash} (q_0, w, w_2 V_2 \alpha_2) \overset{*}{\underset{P}{\vdash}} (q_0, w \setminus w_2, V_2 \alpha_2),$$

  where we let $w \setminus w_2$ to denote the suffix of w_2 in w.
- Now, note that $V_2 \to \gamma_2$ is a valid production rule; hence, there is a valid one-step computation from $(q_0, w \setminus w_2, V_2 \alpha_2)$ that uses the corresponding Type-2 transition. The resultant configuration change will then be

$$(q_0, w, S) \underset{P}{\vdash} (q_0, w, w_2 V_2 \alpha_2) \overset{*}{\underset{P}{\vdash}} (q_0, w \setminus w_2, V_2 \alpha_2) \underset{P}{\vdash} (q_0, w \setminus w_2, (w_3 \setminus w_2) V_3 \alpha_3),$$

  where $(w_3 \setminus w_2) V_3 \alpha_3 := \gamma_2 \alpha_2$.

Explanation for Slide 16 (Continued)

- Again, we see that a portion of the top of the stack contains $w \setminus w_2$, which matches the initial segment of the input tape. Then there is a valid multi-step computation involving $|w_3 \setminus w_2|$ Type-1 transitions that pops $w_3 \setminus w_2$. The resultant configuration will then be $q_0, w \setminus w_3, V_3\alpha_3)$.
- Now, this proceeds until all of w is exhausted (read) from the input tape, and the configuration at the end will be $(q_0, \epsilon, \epsilon)$. Since the stack is empty, the original string w will be accepted.
- $\Leftarrow$ The direction that a trajectory accepting w in P implies a derivation of w in G is simply arguing the above in the reverse direction using the facts that:
  - a trajectory for accepting w in P must consist only of Type-1 and Type-2 transitions, and each Type-2 transition corresponds to a unique production in G.
  - The argument is literally the same as above except that we now uncover the production rule from the corresponding Type-2 transition.

Explanation for Slide 18

Inductive proof for $(q, w, X) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon) \Rightarrow [qXp] \overset{*}{\underset{G}{\Rightarrow}} w$ based on length of computation.

- Basis: Let $(q, w, X) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon)$ be a one-step computation. Thus, w has to be an input symbol or ϵ. Then, by definition of one-step computation it **must** be true that $(p, \epsilon) \in \delta(q, w, x)$, where $X = xX'$. Then, by the construction of G, we have $[qXp] \rightarrow w$ (see Slide 12 for the construction), and hence $[qXp] \overset{*}{\underset{G}{\Rightarrow}} w$.
- Induction: $(q, w, X) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon)$ in say $k > 1$ steps. Let us assume that the in the first step of the computation, the symbol a is read from the input tape (or $a = \epsilon$). Let $w = ax$. Let's break the k-step computation to a single step followed by a $k-1$-step computation as detained in 1 (encircled in black). Let r_1 be the state of the PDA after the first step and let X be popped and $Y_1 \cdots Y_k$ be pushed onto the stack after the first step/transition/move.
- Now, the claim is that the $k-1$ step portion of the computation can be expanded into the sequence of computations as given in 2 (encircled in black). The reasoning is as follows. The ID $(r_1, x, Y_1 \cdots Y_k)$ eventually changes to (p, ϵ, ϵ). There must be a finite number of moves after which the effective stack change is the popping of Y_1, i.e., after a finite number of steps Y_2 is at the top **for the very first time**. The steps until then could have popped Y_1, pushed a string, and then popped it eventually to reveal Y_2 at the top.

Explanation for Slide 18 (Continued)

- Let w_1 be the portion of the input tape read and r_2 be the state pf the PDA when this intermediate ID where Y_2 is at the top of the stack (i.e., the stack contains $Y_2 \cdots, Y_k$) is attained. Thus,

$$(r, x, Y_1 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_2, x \setminus w_1, Y_2, \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon),$$

  where again we let $w \setminus w_1$ to be the suffix of w_1 in w.
- By a similar argument, after reading another segment, say w_2, of the input tape and reaching (some) state r_3, the top of the stack of the PDA contains Y_3 **for the very first time**. Thus,

$$(r, x, Y_1 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_2, x \setminus w_1, Y_2, \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_3, x \setminus (w_1 w_2), Y_3, \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon).$$

- Proceeding inductively, we see that 2 (encircled in black) holds. Note that x is then equal to the concatenation of the w_i's, i.e., $x = w_1 \cdots w_k$.
- Now focus on the computation within the blue block in 2. In no intermediate ID of the computation is Y_2 at the top of the stack (since $(r_2, x \setminus w_1, Y_2, \cdots Y_k)$ is the very first time Y_2 is at the top of the stack). Thus, the stack contents $Y_2 \cdots Y_k$ are never visited in this first set of moves, and hence, we see that

$$(r_1, x, Y_1 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_2, x \setminus w_1, Y_2, \cdots Y_k) \Rightarrow (r_1, w_1, Y_1) \overset{*}{\underset{P}{\vdash}} (r_2, \epsilon, \epsilon). \quad (3)$$

Explanation for Slide 18 (Continued)

- Similarly, we see that the in portion of the computation in orange, no intermediate ID of the computation has Y_3 at the top of the stack (since $(r_3, x \setminus (w_1 w_2), Y_3, \cdots Y_k)$ is the very first time Y_3 is at the top of the stack). Hence,

$$(r_2, x \setminus w_2 \cdots w_k, Y_2, \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_3, w_2 \cdots w_k, Y_3 \cdots Y_k) \Rightarrow (r_2, w_2, Y_2) \overset{*}{\underset{P}{\vdash}} (r_3, \epsilon, \epsilon). \quad (4)$$

- We can proceed inductively to argue that $(r_i, w_i, Y_i) \overset{*}{\underset{P}{\vdash}} (r_{i+1}, \epsilon, \epsilon)$ for $i = 1, \ldots, k-1$.
- Now each of these derivations $(r_i, w_i, Y_i) \overset{*}{\underset{P}{\vdash}} (r_{i+1}, \epsilon, \epsilon)$ for $i = 1, \ldots, k-1$ contain $k-1$ or less steps, because the number of steps they contain is at least one-less than the number of steps in the computation in 1 (encircled in black).
- Consequently, by the induction hypothesis, we have $[r_i Y_i r_{i+1}] \overset{*}{\underset{G}{\Rightarrow}} w_i$, $i = 1, \ldots, k-1$.

  By the very same argument $[r_k Y_k p] \overset{*}{\underset{G}{\Rightarrow}} w_k$.
- Now focus on the yellow box at the top, the first one-step computation guarantees that there exists a production rule

$$[qXp] \to a[r_1 Y_1 r_2][r_2 Y_2 r_3] \cdots [r_{k-1} Y_{k-1} r_k][r_k Y_k p]. \quad (5)$$

  Now combining the above production with the known derivations in 4 (encircled in black), we see that $[qXp] \overset{*}{\underset{G}{\Rightarrow}} aw_1 \cdots w_k = ax = w$.

Explanation for Slide 19

Inductive proof for $(q, w, X) \overset{*}{\underset{P}{\vdash}} (p, \epsilon, \epsilon) \Leftarrow [qXp] \overset{*}{\underset{G}{\Rightarrow}} w$ based on length of leftmost derivation.

- Basis: $[qXp] \overset{*}{\underset{LM}{\Rightarrow}} w$ be a one-step derivation. This can be possible only if $(p, \epsilon) \in (q, w, X)$, which then means $(q, w, X) \underset{P}{\vdash} (p, \epsilon, \epsilon)$.
- Induction: Let $[qXp] \overset{*}{\underset{G}{\Rightarrow}} w$ in $k > 1$ steps. As in the previous direction, let us split the leftmost derivation into the first step and then rest.
- The first step must involve the application of some production rule, say, $[qXp] \rightarrow a[r_0 Y_1 r_1][r_1 Y_2 r_2] \cdots [r_{k-1} Y_k p]$.
- By 1 (encircled in 1) each non-terminal $[r_{i-1} Y_i r_i]$ $i = 1, \ldots, k$ must derive (via a leftmost derivation) a segment of w, say w_i in $k - 1$ steps or less. [w_i is the yield of the parse subtree in the parse tree of $[qXp]$ with yield w, and the depth of the subtree is at most 1 less than the depth of the parse tree of $[qXp]$.).
- Hence, $[r_{i-1} Y_i r_i] \overset{*}{\underset{LM}{\Rightarrow}} w_i$ for $i = 1, \ldots, k$ in $k - 1$ steps or less (I've set $r_k = p$ here).

  By induction hypothesis, then $(r_{i-1}, w_i, Y_i) \overset{*}{\underset{P}{\vdash}} (r_i, \epsilon, \epsilon)$.
- Then by Lemma 6.2.1, $(r_{i-1}, w_i \cdots w_k, Y_i \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_i, w_{i+1} \cdots w_k, Y_{i+1} \cdots Y_k)$. Thus,

$$(q, w, X) \underset{P}{\vdash} (r_0, w_1 \cdots w_k, Y_1 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_1, w_2 \cdots w_k, Y_2 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (r_k, \epsilon, \epsilon) = (p, \epsilon, \epsilon).$$