COMP3630 / COMP6363

week 5: **Introduction to Turing Machines**

This Lecture Covers Chapter 8 of HMU: Introduction to Turing Machines

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convenor & lecturer: Pascal Bercher

The Australian National University

Semester 1, 2025

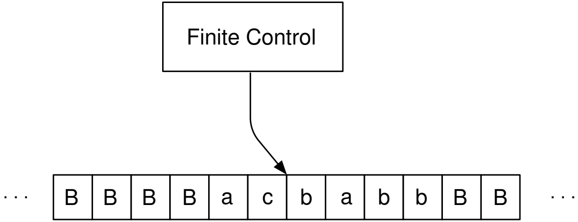
Content of this Chapter

- Turing Machine
- Extensions of Turing Machines
- Restrictions of Turing Machines
- Extensions of PDAs – and Relationship to TMs

Additional Reading: Chapter 8 of HMU.

Introduction to TMs

Turing Machine: Informal Definition



- A tape extending infinitely in both sides
- A reading head that can edit tape, move right or left
- A finite control
- A string is accepted if finite control ever reaches a final/accepting state

Heads-up: There are many variations of TMs (e.g., in COMP1600, the head could also stay stationary), and we will go through a few of them.

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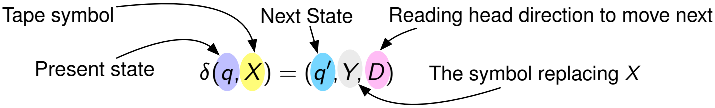
- δ: (deterministic) transition function. δ is a **partial function** over $Q \times \Gamma$, where the first component is viewed as the present state, and the second is viewed as the tape symbol read. If $\delta(q, X)$ is defined, then



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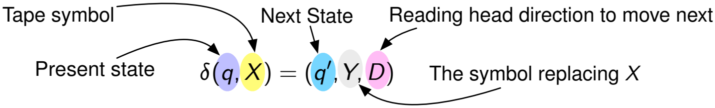

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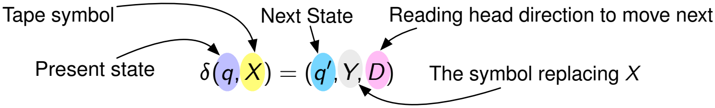

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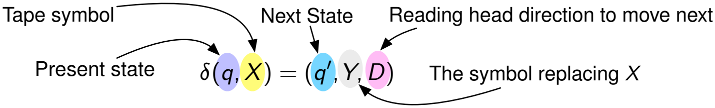

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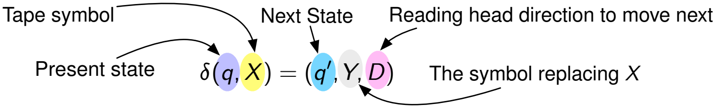

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- Head **always** moves to the left or right. Being stationary is not an option. It can also be defined with such an option, see tutorial.

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- Turing machines can be defined by describing δ using a transition table.
- They can also be defined using transition diagrams (with labels appropriately altered)

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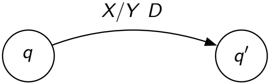

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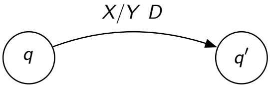

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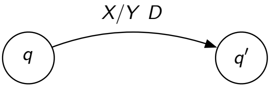

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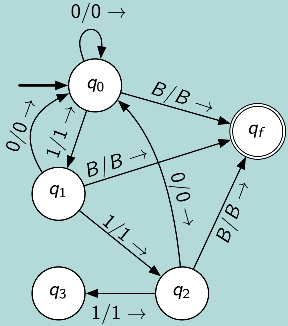

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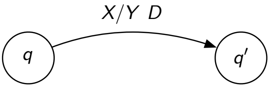

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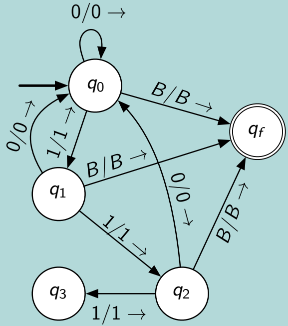

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Because we never manipulate the tape and terminate once the String is read. The only difference is that not all edges are defined, but this can be fixed with a trap state.

Instantaneous Descriptions of TMs

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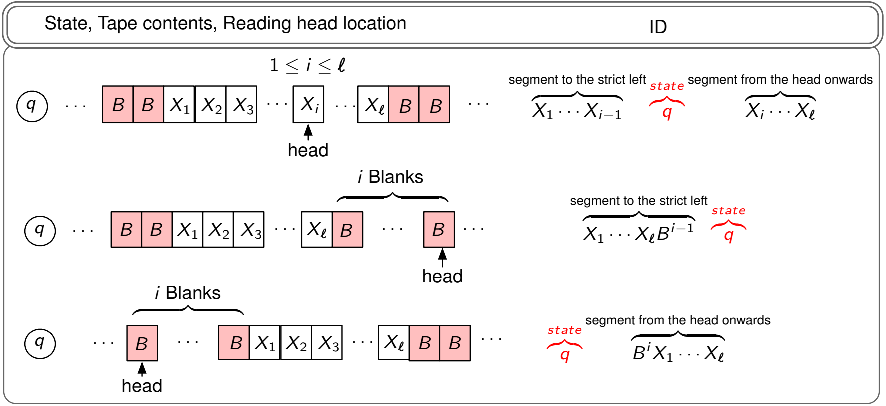
- The instantaneous description or configuration or ID of a TM contains 3 parts:
  (a) The (finite, non-trivial) portion of tape to the left of the reading head;
  (b) the state that the TM is presently in; and
  (c) the (finite, non-trivial) portion of the tape to the right of the reading head.



'Moves' of a TM

- Just as in the case of a PDA, we use $\underset{M}{\vdash}$ to indicate a single move of a TM M, and $\overset{*}{\underset{M}{\vdash}}$ to indicate zero or a finite number of moves of a TM.

Present ID	Transition	**Next** ID
$X_1 \cdots X_{i-1} q X_i \cdots X_\ell$	$\delta(q, X_i) = (q', Y, R)$	$X_1 \cdots X_{i-1} Y q' X_{i+1} \cdots X_\ell$
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$q B^i X_1 \ldots X_\ell$	$\delta(q, B) = (q', Y, R)$	$\begin{cases} Y q' X_2 \cdots X_\ell & i = 0 \\ Y q' B^{i-1} X_1 \cdots X_\ell & i > 0 \end{cases}$
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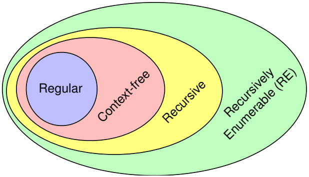
- A language L is **recursive** if it is accepted by a TM that **always** halts on all inputs.



(Context-sensitive languages sit between the CFLs and recursive languages.)

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- Note: Just because $L \in RE$ does not mean $L \notin R$ since $R \subseteq RE$. Also, $R \subsetneq RE$.

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- "Accepting w", i.e. $w \in L(M)$, does not imply halting, since the machine may keep going after reaching a final state. If we design a TM, there's no point in defining transitions out of any final state (so, it would halt).
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On Acceptance, Rejection, Halting, and Decidability

	Accepted $w \in L(M)$	**Rejected** $w \notin L(M)$
Halted on *w*		
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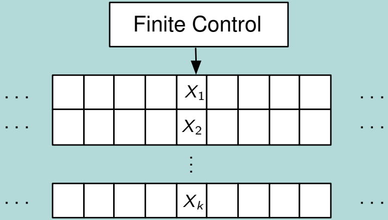
Extensions of TMs

Multiple-Track TMs

Multiple-track TM

- We do not provide a formal definition (but assume you could provide one).
- There are k tracks, each having symbols written on them. They are essentially tapes, but we call them that way since they are not independent.



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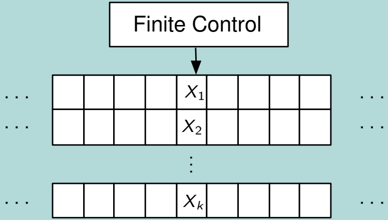
- Likewise, all tapes move simultaneously in the same direction.



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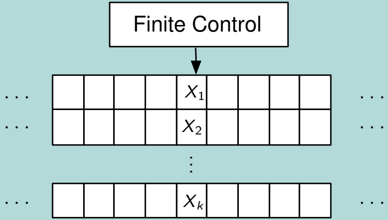

- A k-track TM with tape alphabet Γ has the same language-acceptance power as a TM with tape alphabet Γ^k. (E.g., each cell contains the "symbol" $(X_1, \ldots, X_k)$)

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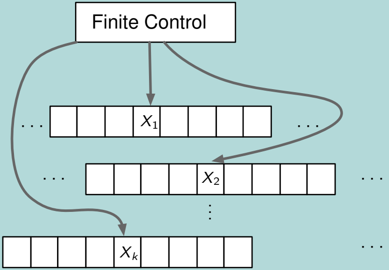

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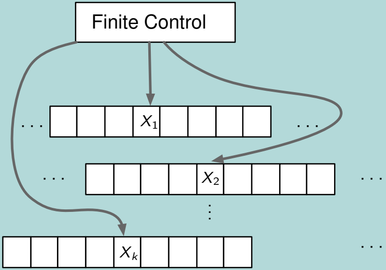

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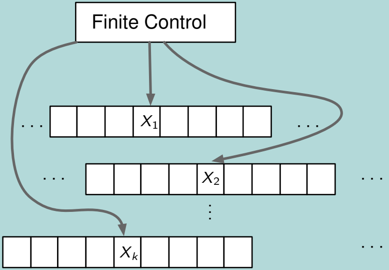

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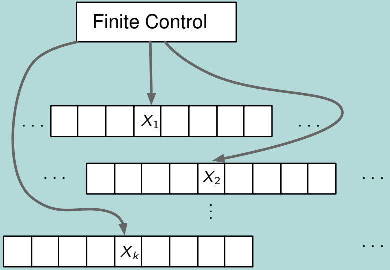

- The rest stays the same (e.g., one set of states, acceptance, etc.).

Multi-tape TMs

Theorem 8.2.1

Every language that is accepted by a multi-tape TM is also recursively enumerable (i.e., accepted by some 'standard' TM).

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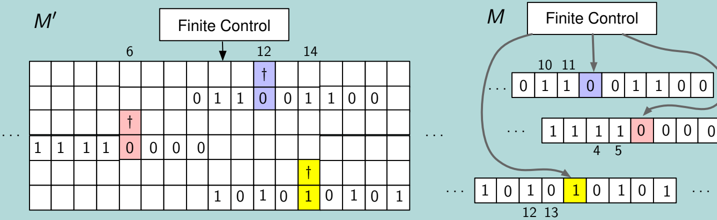

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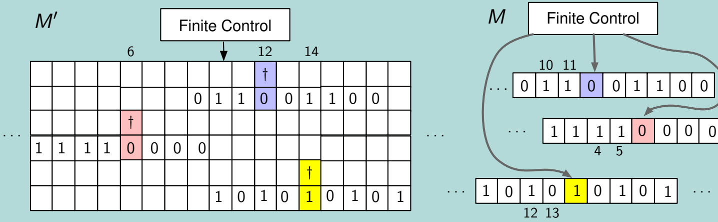
- Every odd track has an alphabet $\{B, \dagger\}$, and contains a single $\dagger$.
  The $2i-1^{\text{th}}$ track of M' contains $\dagger$ at the location where the i^{th} head of M is located.

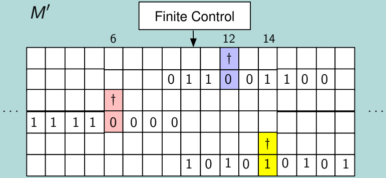


Multi-tape TMs

Proof of Theorem 8.2.1 (1 of 3)

What is the main problem we need to solve?

- In the Multi-tape TM M, heads move independently, whereas in the Multi-track TM M' they do not. So the heads can diverge:



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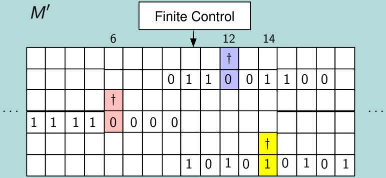

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So, how to solve it?

- Make sure that in each transition of M, we visit all heads of M'.
- "Store" all head positions in a state with k (number of tapes) entries.

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Proof of Theorem 8.2.1 (2 of 3)

- The state of M' has 3 components: (a) the state of M; (b) the number of †s to its head's strict left; and (c) a k-length tuple from $(\Gamma \cup \{?\})^k$.

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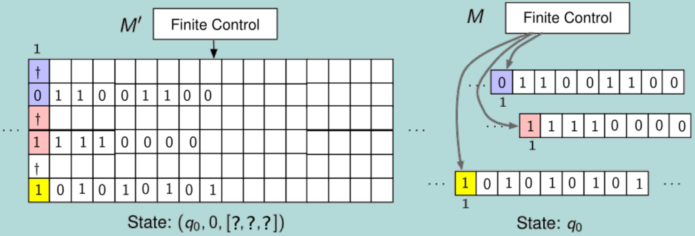

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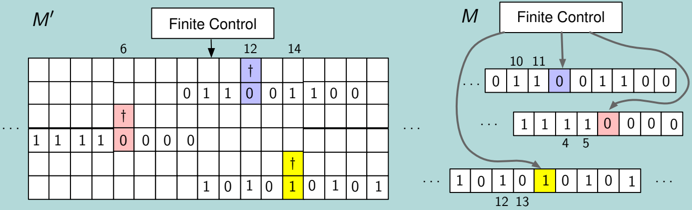
- Each move of M takes multiple moves of M', and is a sweep of the tape from the location of the leftmost † to that of the rightmost † and back performing the changes to tracks that M would do to its corresponding tapes.
- The right sweep ends when the second component is k.



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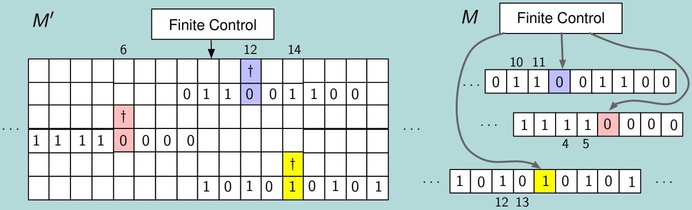

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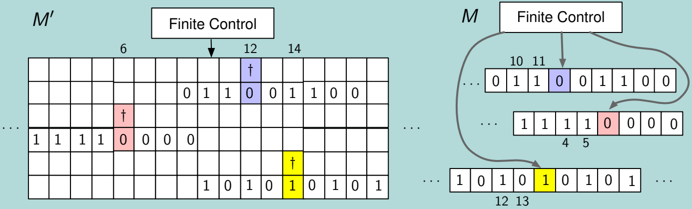

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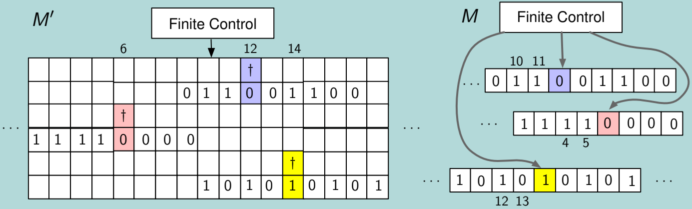

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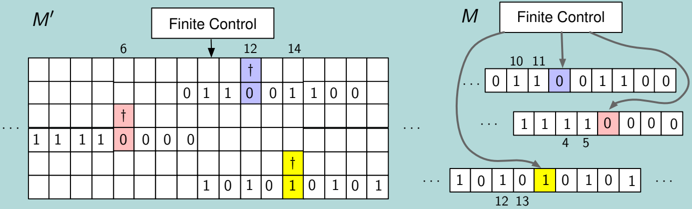

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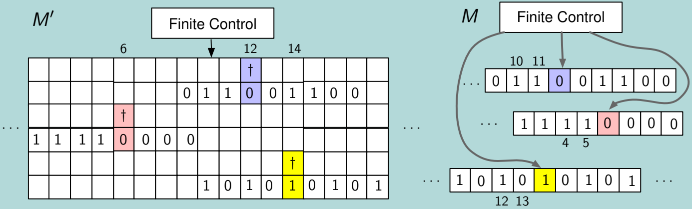
- Note that M' mimics M and hence the languages accepted are identical.



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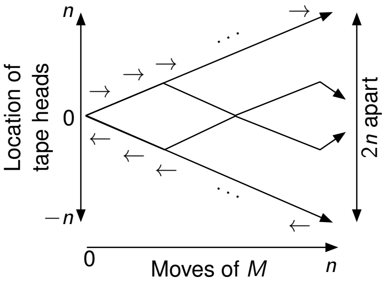

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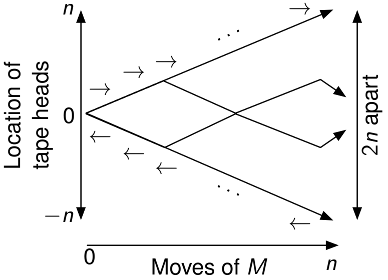

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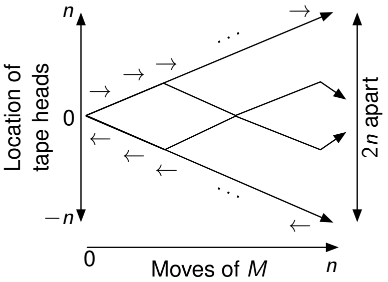

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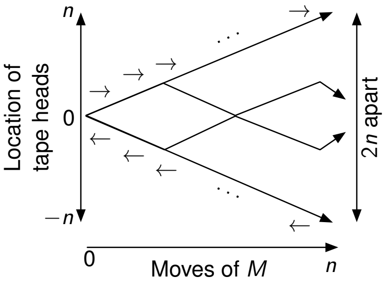
- So, n moves in M need $O(n^2)$ moves in M'.



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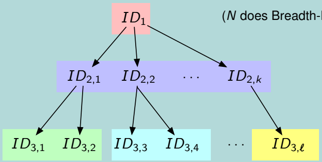

(N does Breadth-First exploration of IDs of M)

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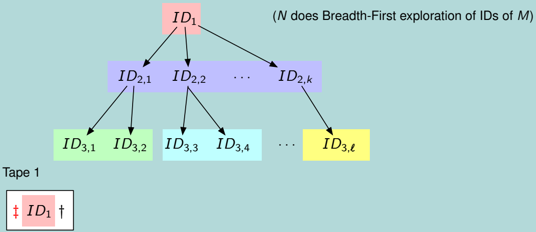

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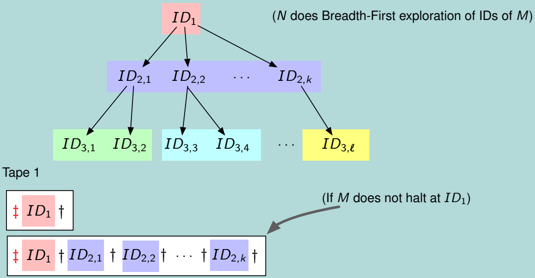

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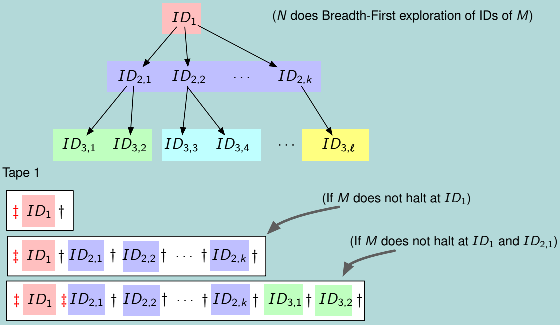

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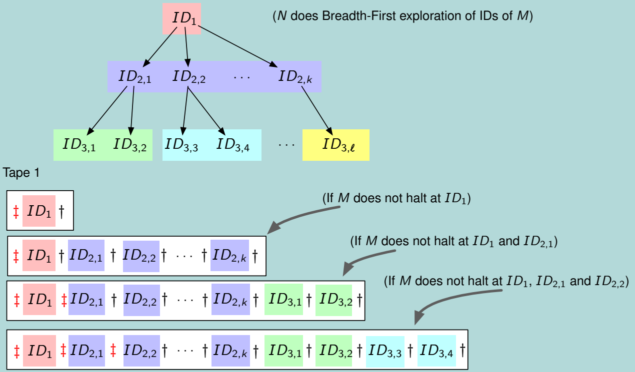
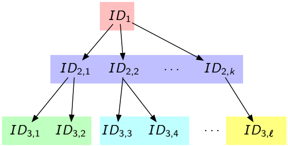
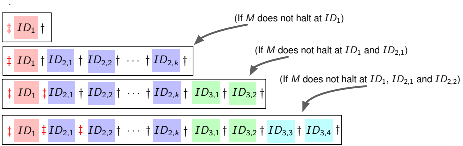


Outline of Proof of Theorem 8.2.3

- We can devise a 2-tape TM N that simulates M.
- N first replaces the content of the first tape by ‡ followed by the ID that M is initially in, which is then followed by a special symbol †, which serves as ID separator. (N uses the second tape as scratch tape to enable this operation).
- If the ID corresponds to a final state, N accepts (as would M).
- If not, N then identifies all possible choices for the next IDs for M and enters each one of them followed by † at the right end of its first tape. (Again, N uses the second tape as scratch tape to enable this operation.)
- N then searches for † to the right of ‡, changes the † to a ‡ (to signify that it is processing the succeeding ID), and processes that ID in the similar way.
- N halts at an ID iff M would at that ID. (To have halting equivalence.)





Restrictions of TMs

TM Semi-infinite Tape

A TM with a semi-infinite tape is a TM that only has blanks on one of its sides, but not on the other.

Phrased (slightly) more formally:
A TM with a semi-infinite tape is a TM that can never move to left of the left-most input symbol.

We don't provide a formal definition, but a way of simulating this is by providing a special symbol, placed on the left of the input, and defining the transitions to always go to the right when this is read.

TM Semi-infinite Tape

Theorem 8.3.1

Every recursively enumerable language is also accepted by a TM with semi-infinite tape.

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- Given a TM M that accepts a language L, construct a two-track TM M' as follows.
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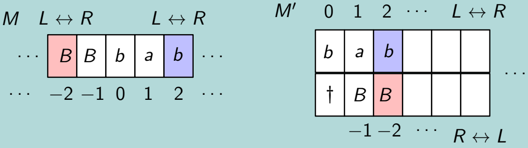

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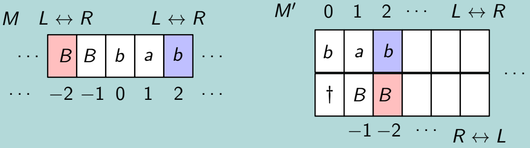

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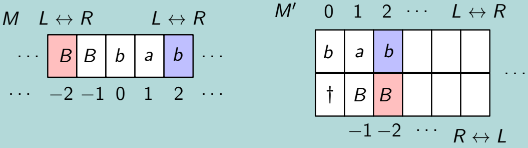

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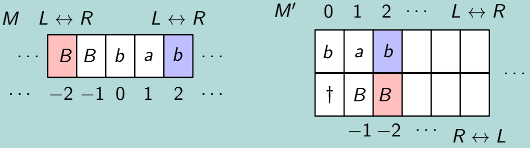
- At any time, M' keeps track of whether M is to the right or left of its start location.
  - If M is to the strict right of its start location, M' mimics M on the first track.
  - If M is to the strict left of its start location, M' mimics M on second track, but with the head directions reversed. M' detects the start by the † symbol.



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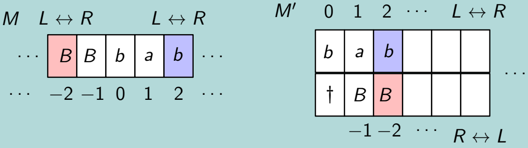
- It can be formally shown that M' accepts a string iff M accepts it.



Extensions of PDAs

Multi-stack Machines

A multistack machine is a PDA with several independent stacks (i.e., one can be popping a symbol, while another is pushing a symbol).

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Theorem 8.4.1

Every recursively enumerable language is accepted by a two-stack PDA

Outline of Proof of Theorem 8.4.1

- Let each stack again contain a bottom-most start symbol.
- Let ID $= x_{-3}x_{-2}x_{-1}qx_0x_1x_2$, i.e., $w = x_{-3}x_{-2}x_{-1}x_0x_1x_2$, and head read reads x_0
  - Let stack-1 contain $x_0x_1x_2$ (with x_0 at the top), representing the head position and the symbols to its right.
  - Let stack-2 contain $x_{-1}x_{-2}x_{-3}$ (with x_{-1} at the top), representing the symbols to the left of the head in reversed order.

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  - Let stack-2 contain $x_{-1}x_{-2}x_{-3}$ (with x_{-1} at the top), representing the symbols to the left of the head in reversed order.
- What if we move the head to the right? Then, ID' $= x_{-3}x_{-2}x_{-1}x_0q'x_1x_2$.
  We can easily do this with our stacks:
  - How should the stack now look like?
  - stack-1: x_1x_2 and stack-2: $x_0x_{-1}x_{-2}x_{-3}$.
  - But that's just a simple pop and push!
- Moving to the left, and changing the symbol that's written can be simulated as well.

Multi-stack Machines

Outline of Proof of Theorem 8.4.1, cont'd

- Remaining problem: How to fill the stacks initially?
- Recall: stack-1 contains what's right of the head and stack-2 what's left (but reversed).

Multi-stack Machines

Outline of Proof of Theorem 8.4.1, cont'd

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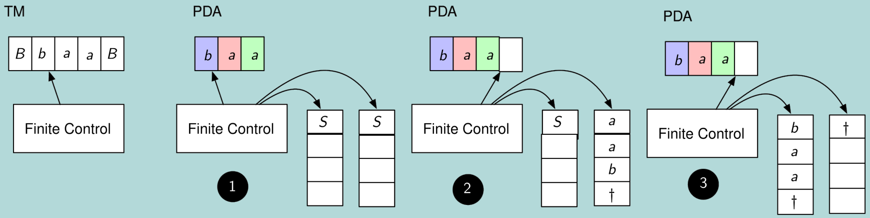
Multi-stack Machines

Outline of Proof of Theorem 8.4.1, cont'd

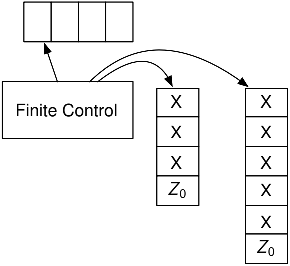
- Remaining problem: How to fill the stacks initially?
- Recall: stack-1 contains what's right of the head and stack-2 what's left (but reversed).
- Initial configuration is $q_0 w$, so stack-1 should be w and stack-2 "empty".
- We achieve this by the following procedure:



- I.e., run to the right filling stack-2, then run back putting it on stack-1.

Counter Machines

- A counter machine is a multi-stack machine whose stack alphabet contains two symbols: Z_0 (stack end marker) and X
- Z_0 is initially in the stack.
- Z_0 may be replaced by $X^i Z_0$ for some $i \geq 0$
- X may be replaced by X^i for some $i \geq 0$.



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- Z_0 may be replaced by $X^i Z_0$ for some $i \geq 0$
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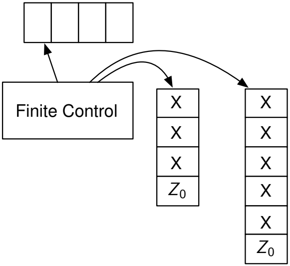
- A counter machine effectively stores a non-negative number.



Simulating a 2-Stack PDA with a 3-Counter Machine

Theorem 8.4.2

Every recursively enumerable language is accepted by a three-counter machine.

Key Challenge

- Challenges:
  - A 2-stack PDA uses arbitrary symbols on its stacks.
  - A counter machine can only store and manipulate numbers.
  - We must encode a stack's contents into a single number so that counter operations can simulate stack operations.
  - We must implement mathematical operations in a stack to encode push and pop operations on the "single numbers".
- How stacks work?
  - counter 1 and 2 encode stacks 1 and 3
  - counter 3 is used for additional computations

Encoding a Stack into a Counter

Encoding Method

- Assign each symbol in the stack alphabet a unique number:
  - $A = 0, B = 1, C = 2, \ldots, D = 3$, etc.
- Represent a stack as a single number using positional encoding:

$$X = Y_1 + rY_2 + r^2 Y_3 + \cdots + r^{k-1} Y_k$$

  where:
  - Y_1 is the top symbol,
  - $Y_2, Y_3, \ldots$ are symbols below it,
  - $r = |\Gamma|$, the size of the stack alphabet.

Example

- Suppose the stack contains (top to bottom): B, C, A, A, D.
- Let $r = 4$ (since the alphabet has 4 symbols).
- Encode as: $X = 1 + 4(2) + 4^2(0) + 4^3(0) + 4^4(3) = 777$.
- The counter now stores $X = 777$.

Simulating Stack Operations

Popping the Top Symbol

- Extract the top symbol using $X \mod r$:

$$777 \mod 4 = 1 \quad \Rightarrow \quad \text{Top symbol was } B$$

- Remove it by dividing by r: $X' = \lfloor X/4 \rfloor = 194$.
- The counter now stores $X' = 194$, encoding stack $C, A, A, D = 2 + 4(0) + 4^2(0) + 4^3(3) = 194$.

Pushing a Symbol

- Suppose we push symbol A onto the stack C, A, A, D.
- The current stack encoding is: $X = 194$.
- Compute the new encoding:

$$X' = 4 \cdot X + 0 = 4 \cdot 194 + 0 = 776.$$

- The counter now stores $X' = 776$, encoding stack A, C, A, A, D:

$$776 = 0 + 4(2) + 4^2(0) + 4^3(0) + 4^4(3).$$

Counter Machine Implementation

- Computing $X \mod r$ (extracting top symbol from stack 1 or 2):
  - Subtract r repeatedly from the respective stack counter (1 or 2) until value is less than r.
  - This remaining value is the top symbol.
- Computing $X' = \lfloor X/r \rfloor$ (removing top symbol from stack 1 or 2):
  - Move remainder to counter 3.
  - Subtract remainder from stack counter.
  - Divide stack counter by r by decrementing it while incrementing counter 3 in steps of r.
- Computing $X' = rX + Z$ (pushing a symbol onto a stack):
  - Copy X to counter 3.
  - Multiply counter 3 by r by adding it to itself r times.
  - Add Z to counter 3.
  - Copy the result back to the respective stack counter (1 or 2).
- Ensuring correct stack operations:
  - Stack 1 and stack 2 each use a separate counter.
  - When operating on stack 1, stack 2 stays unchanged, and vice versa.
  - Counter 3 is used only as temporary storage.

Simulating a 3-Counter Machine with 2 Counters

Theorem 8.4.3

Every recursively enumerable language is accepted by a two-counter machine.

Key Idea

- Encode three counters using prime factorization:

$$X = 2^i 3^j 5^k, \tag{1}$$

  where i, j, k are the values of the three counters.
- Updates involve:
  - Multiplying by 2, 3, or 5 to increment.
  - Dividing by 2, 3, or 5 to decrement.
  - Checking divisibility to test for zero.
- Since a second counter can store temporary results, a 2-counter machine can simulate a 3-counter machine.