COMP3630 / COMP6363

week 6: **Decidability and Undecidability**

This Lecture Covers Chapter 9 of HMU: Decidability and Undecidability

slides created by: Dirk Pattinson, based on material by
Peter Hoefner and Rob van Glabbeck; with improvements by Pascal Bercher

convenor & lecturer: Pascal Bercher

The Australian National University

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Content of this Chapter

- Preliminary Ideas
- Example of a non-RE language
- Recursive languages
- Universal Language
- Reductions of Problems
- Rice's Theorem
- Post's Correspondence Problem
- Undecidable Problems about CFGs

Additional Reading: Chapter 9 of HMU.

Preliminary Ideas

Enumeration of (Binary) Strings

- We can construct a bijective map ϕ from the set of binary strings $\{0,1\}^*$ to natural numbers $\mathbb{N}$.
  - Why might that appear surprising?
  - Because each number has a unique binary encoding, but for each we could add an arbitrary number of zeros in the front, so there seem to be more strings over $\{0,1\}$ than numbers in $\mathbb{N}$.

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- Enlist all strings ordered by length, and for each length, order using lexicographic ordering.

w	$\phi(w)$
ϵ	0
0	1
1	2
00	3
01	4
10	5
11	6
000	7
$\vdots$	$\vdots$
111	16
0000	17
$\vdots$	$\vdots$
1111	32
$\vdots$	$\vdots$

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- Enlist all strings ordered by length, and for each length, order using lexicographic ordering.
- The set of finite binary strings is countable/denumerable.

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A Code for Turing Machines

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- Consider $M = (Q, \Sigma = \{0, 1\}, \Gamma = \{0, 1, B, X_4, \ldots, X_\ell\}, \delta, q_1, B, F)$.
  - Rename states $\{q_1, \ldots, q_k\}$ for $k = |Q|$ with q_1: start state and q_k: final state.
  - Rename input alphabet using $X_1 = 0$, $X_2 = 1$, and blank B as X_3.
  - Rename the rest of the tape symbols by $X_4, \ldots, X_\ell$ for $\ell = |\Gamma|$.
  - Rename L as D_1 and R and D_2. (The directions.)

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- Every transition $\delta(q_i, X_j) = (q_k, X_l, D_m)$ can be represented as a tuple (i, j, k, l, m).

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- Every transition $\delta(q_i, X_j) = (q_k, X_l, D_m)$ can be represented as a tuple (i,j,k,l,m).
- Map each transition tuple (i,j,k,l,m) to a **unique** binary string $0^i 1 0^j 1 0^k 1 0^l 1 0^m$.
  NB: No string representing a transition tuple contains 11.

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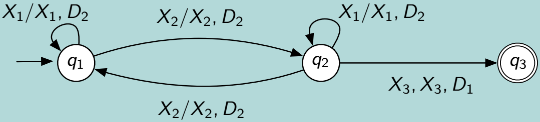
- Every transition $\delta(q_i, X_j) = (q_k, X_l, D_m)$ can be represented as a tuple (i,j,k,l,m).
- Map each transition tuple (i,j,k,l,m) to a **unique** binary string $0^i10^j10^k10^l10^m$. NB: No string representing a transition tuple contains 11.
- Order transition tuples lexicographically and concatenate all transitions using 11 to indicate end of a transition. Let the resultant string be w_M. For example, 3 transitions can be combined as $\underbrace{0^{i_1}10^{j_1}10^{k_1}10^{l_1}10^{m_1}}_{\text{1st transition}}\,11\,\underbrace{0^{i_2}10^{j_2}10^{k_2}10^{l_2}10^{m_2}}_{\text{2nd transition}}\,11\,\underbrace{0^{i_3}10^{j_3}10^{k_3}10^{l_3}10^{m_3}}_{\text{3rd transition}}$
- For each TM M, define the code $\langle M\rangle$ for TM M as w_M.

The Set of Turing Machines

An Example: A TM that accepts strings with odd # of 1s



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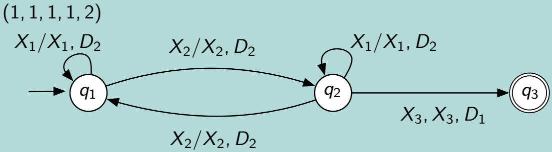

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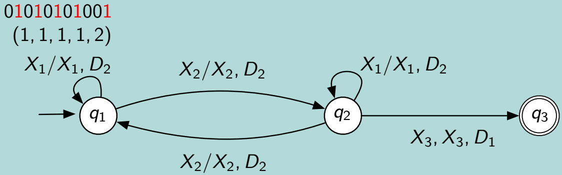

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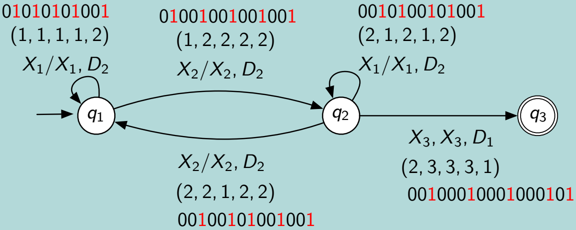

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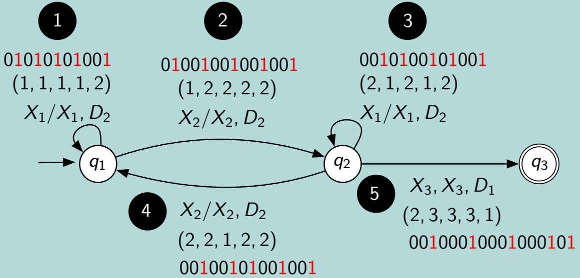

$$\langle M \rangle = (01)^4 0^2 111 010^2 10^2 10^2 10^2 111 0^2 1010^2 1010^2 111 0^2 10^2 1010^2 10^2 111 0^2 10^3 10^3 10^3 101.$$

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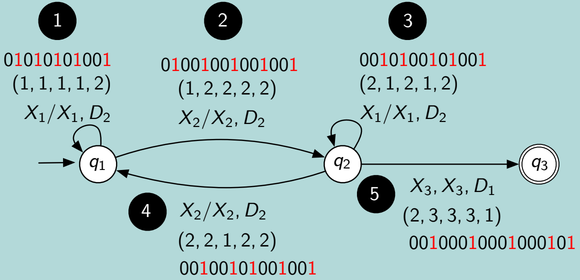

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Remark 9.1.1

- *Each TM M encoding has a unique natural number, i.e., $\phi(\langle M\rangle)$; Each TM M may have several codes $\langle M\rangle$ and thus several numbers; but each natural number corresponds to* **at most** *one TM.*

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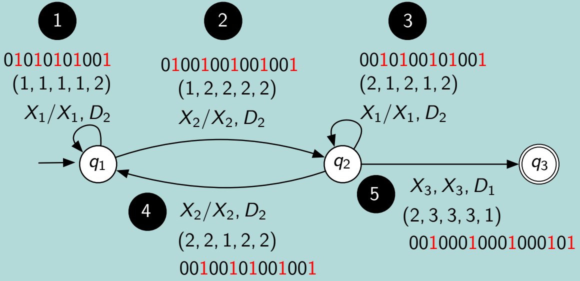

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  Each TM M may have several codes $\langle M\rangle$ and thus several numbers;
  but each natural number corresponds to **at most** *one TM.*
- *The set of TMs/RE languages/CSLs/CFLs/regular languages is countable, i.e., finite or there is a bijection to the natural numbers. Careful: Countable $\not\Rightarrow$ RE membership! Clear, since every language is countable, but some are not in RE.*

Example of a non-RE language

First, a recap:

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- $L \in RE$: We know that there is a TM M with $L(M) = L$. (Recall that we can assume that it terminates on words in L.) However, for any $w \notin L$, M might not terminate. This means that we are only sure to learn about membership, but non-membership may be "stated" only sometimes. What's the worst about that?!

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- $L \notin RE$: There does exist a TM with $L(M) = L$! So, what if we attempted to write a TM <u>anyways</u>? It will be "wrong"! E.g.,
  - for some $w \in L$, it will reject, i.e., without accepting, terminate or loop forever.
  - for some $w \notin L$, it will accept it.

Now: We see the most famous language L with $L \notin RE$.

Diagonalization Language L_d

- Fix ϕ as on slide 4. Now, for each w_i (the i^{th} string in our enumeration) define an M_i: If w_i encodes a TM, take it! Otherwise, define it as a trivial one with empty language. Thus, we get $\phi(\langle M_i \rangle) = i$ for all $i \in \mathbb{N}$, where most M_is are artificial/trivial, but we list all 'actual' ones!

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- Construct an infinite table. Rows: M_0, M_1, ... as above and cols: All Strings according to slide 4. Cell $(i, j) = 1$ iff M_i accepts $w_j := \phi^{-1}(j)$.

I.e., $\phi(\epsilon) = 0, \phi(0) = 1,$ $\phi(1) = 2, \phi(00) = 3, \ldots$ just like on page 4!

E.g., $\phi(w_2) = 2$ with $0, 1, 00, 11 \in L(M_2)$, but is that true? No! $\phi^{-1}(2) = 1$, i.e., $\phi(1) = 2$, but 1 does not encode a TM, so M_2 is a trivial one with $L(M_2) = \emptyset$

So, this table is just an illustration!

	ϵ $\phi^{-1}(0)$	0 $\phi^{-1}(1)$	1 $\phi^{-1}(2)$	00 $\phi^{-1}(3)$	01 $\phi^{-1}(4)$	10 $\phi^{-1}(5)$	11 $\phi^{-1}(6)$	...
ϵ=M_0	**0**	0	0	0	0	0	0	
0=M_1	1	**1**	0	0	0	1	1	
1=M_2	0	1	**1**	1	0	0	1	
00=M_3	1	1	1	**0**	0	1	1	
01=M_4	1	0	0	1	**1**	0	0	
10=M_5	1	1	0	0	0	**0**	1	
$\vdots$								

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- Construct an infinite table. Rows: M_0, M_1, ... as above and cols: All Strings according to slide 4. Cell $(i, j) = 1$ iff M_i accepts $w_j := \phi^{-1}(j)$.
- Define a language $L_d = \{w_j : M_j \text{ does not accept } w_j, \text{ where } j \in \mathbb{N}\}$.

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$\epsilon = M_0$	**0** ✔	**0**	**0**	**0**	**0**	**0**	**0**	
$0 = M_1$	**1**	**1**	**0**	**0**	**0**	**1**	**1**	
$1 = M_2$	**0**	**1**	**1**	**1**	**0**	**0**	**1**	
$00 = M_3$	**1**	**1**	**1**	**0** ✔	**0**	**1**	**1**	
$01 = M_4$	**1**	**0**	**0**	**1**	**1**	**0**	**0**	
$10 = M_5$	**1**	**1**	**0**	**0**	**0**	**0** ✔	**1**	
$\vdots$								

$L_d = \{\epsilon, 00, 10, \ldots\}$ i.e., all TMs that do not accept their own encoding.

Because $\phi^{-1}(0) = \epsilon \notin L(M_0)$, $\phi^{-1}(3) = 00 \notin L(M_3)$, etc.

L_d is not a recursively enumerable language

- L_d cannot be accepted by **any** TM. Proof by contradiction.

	✔ ϵ $\phi^{-1}(0)$	0 $\phi^{-1}(1)$	1 $\phi^{-1}(2)$	✔ 00 $\phi^{-1}(3)$	01 $\phi^{-1}(4)$	✔ 10 $\phi^{-1}(5)$	11 $\phi^{-1}(6)$	...
$\epsilon = M_0$	**0** ✔	**0**	**0**	**0**	**0**	**0**	**0**	
$0 = M_1$	**1**	**1**	**0**	**0**	**0**	**1**	**1**	
$1 = M_2$	**0**	**1**	**1**	**1**	**0**	**0**	**1**	
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- L_d cannot be accepted by **any** TM. Proof by contradiction.
- Assume it were. Then there is a TM M_j accepting L_d, i.e., $L(M_j) = L_d$.

	✓			✓		✓		
	ϵ	0	1	00	01	10	11	...
	$\phi^{-1}(0)$	$\phi^{-1}(1)$	$\phi^{-1}(2)$	$\phi^{-1}(3)$	$\phi^{-1}(4)$	$\phi^{-1}(5)$	$\phi^{-1}(6)$	
$\epsilon = M_0$	**0** ✓	**0**	**0**	**0**	**0**	**0**	**0**	
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$1 = M_2$	**0**	**1**	**1**	**1**	**0**	**0**	**1**	
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- L_d cannot be accepted by **any** TM. Proof by contradiction.
- Assume it were. Then there is a TM M_j accepting L_d, i.e., $L(M_j) = L_d$.
- But now we get a contradiction:
  - If $(j, j) = 1$, then $w_j \in L(M_j)$ by definition of $(j, j) = 1$.
    But if $w_j \in L(M_j)$, then $w_j \notin L_d$, so cell (j, j) should be 0! $\lightning$

	✔			✔		✔		
	ϵ	0	1	00	01	10	11	...
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L_d is not a recursively enumerable language

- L_d cannot be accepted by **any** TM. Proof by contradiction.
- Assume it were. Then there is a TM M_j accepting L_d, i.e., $L(M_j) = L_d$.
- But now we get a contradiction:
  - If $(j, j) = 1$, then $w_j \in L(M_j)$ by definition of $(j, j) = 1$.
    But if $w_j \in L(M_j)$, then $w_j \notin L_d$, so cell (j, j) should be 0! ↯
  - If $(j, j) = 0$, then $w_j \notin L(M_j)$ by definition of L_d.
    But if $w_j \notin L(M_j)$, then $w_j \in L_d$, so cell (j, j) should be 1! ↯

	✔ ϵ $\phi^{-1}(0)$	0 $\phi^{-1}(1)$	1 $\phi^{-1}(2)$	✔ 00 $\phi^{-1}(3)$	01 $\phi^{-1}(4)$	✔ 10 $\phi^{-1}(5)$	11 $\phi^{-1}(6)$	...
$\epsilon = M_0$	**0** ✔	0	0	0	0	0	0	
$0 = M_1$	1	**1**	0	0	0	1	1	
$1 = M_2$	0	1	**1**	1	0	0	1	
$00 = M_3$	1	1	1	**0** ✔	0	1	1	
$01 = M_4$	1	0	0	1	**1**	0	0	
$10 = M_5$	1	1	0	0	0	**0** ✔	1	
$\vdots$								

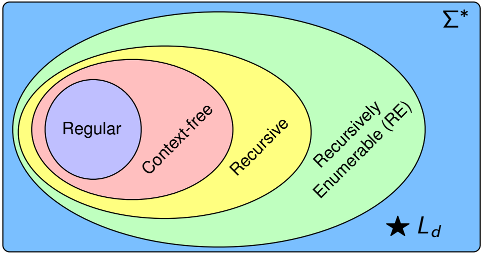
$L_d = \{\epsilon, 00, 10, \ldots\}$

Recursive languages

Recursive Languages

Recall the following definitions:

- A language L is **recursive** if it is accepted by a TM M that halts on **all** inputs
  - In such a case, the TM M is said to **decide** L.
  - Every recursive language is recursively enumerable (by definition).



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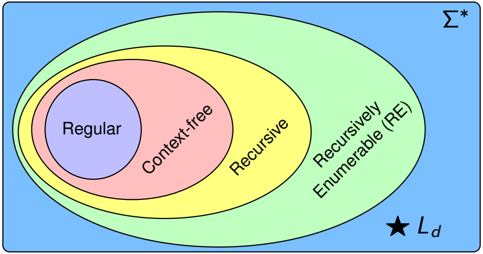


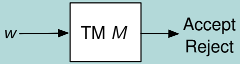
- Do not confuse deciding with accepting! TMs can accept without always terminating (namely, e.g, for languages in $RE \setminus R$, where R denotes the recursive languages).

(Some Obvious) Properties of Recursive Languages

Theorem 9.3.1

If L is **recursive**, *so is L^c.*

Proof of Theorem 9.3.1



- Note that M always halts.

- Recursive languages are closed under complementation.

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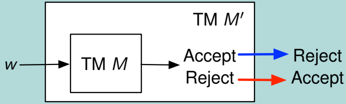

- Note that M always halts. M' does too.
- Accepting states of M with $L(M) = L$ are non-accepting states of M' with $L(M') = L^c$.
- Add a new and only final state q_f in M' such that:

$$\delta_M(q, X) \text{ undefined and } q \notin F$$
$$\text{(I.e., } M \text{ rejects in } q \text{ for } X\text{)}$$
$$\Downarrow$$
$$\delta_{M'}(q, X) = (q_f, X, R).$$
$$\text{(I.e., } M' \text{accepts in that case)}$$

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(Some Obvious) Properties of Recursive Languages

Theorem 9.3.2

If L and L^c are both recursively enumerable, then L (and hence L^c) is (are) recursive.

Proof of Theorem 9.3.2

- Let $L = L(M_1)$ and $L^c = L(M_2)$. (By definition, M_1 and M_2 must exist!)
- Simulate running M_1 and M_2 in parallel by using a 2-tape TM M. M's states (q, q') use q from M_1 and q' from M_2.
- Declare final state of M is q is final in M_1. If M_2 rejects, then M accepts.
- Continue running M until a final state is reached.
- Since for any word either sub machine will halt, M will terminate and accept L.

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Alternate Definition of Recursive Languages

L is recursive if both L and L^c are recursively enumerable.

The Universal Language and Turing Machine

Intermission

To what does a TM (somehow) correspond to?

- To a (general purpose) computer?
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- To a (general purpose) computer?
- Or to a (specific) program?

Any TM is like a program! Because it does one job and one job only! Your computer, on the other hand, can take any job (by loading different programs)!

So, how could we generalize that?

In a nutshell,
A TM is called universal TM iff it can simulate any TM.

The Universal Language and Turing Machine

Universal Language L_u

- $L_u := \{\langle M\rangle 111 w : \langle M\rangle$ encodes TM M and $w \in L(M)\}$. [See Slide 4]

Universal TM U (modelled as 5-tape TM)

The Universal Language and Turing Machine

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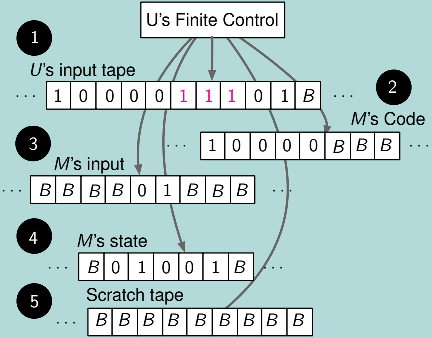
1. U copies $\langle M \rangle$ to tape 2 and verifies it for valid structure.



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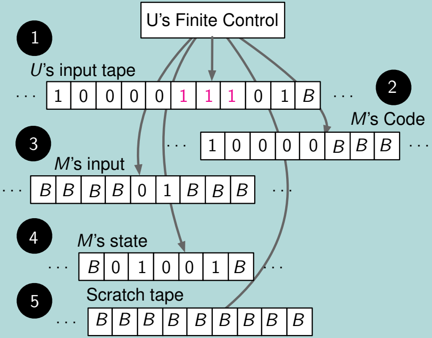

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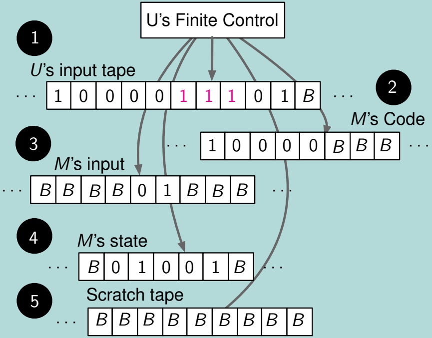

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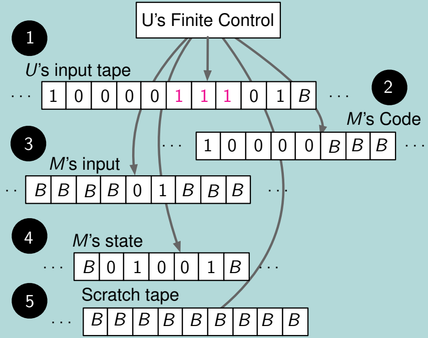
4 To simulate a move of M, U reads tapes 3 and 4 to identify M's state and input as 0^i and 0^j; if state is accepting, M (and hence U) accepts its inputs and halts. Else, U scans tape 2 for 110^i10^j1 or $BB0^i10^j1$.

- If found, using the transition, tapes 4 and 3 are updated, and tape 3's head moves to right or left.
- If not, M halts, and so does U.



Where does L_u Lie in the Hierarchy of Languages?

Theorem 9.4.1

L_u is recursively enumerable, but is not recursive.

Proof of Theorem 9.4.1

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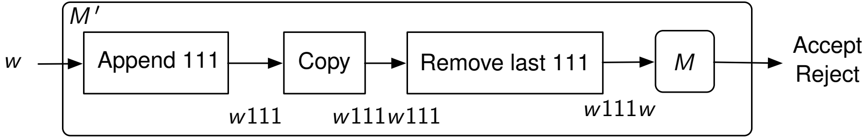

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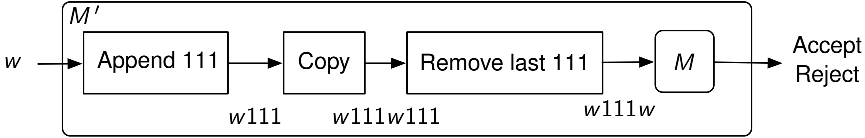
- M' accepts $w \iff w111w \in L_u^c$ Why? Since $w111w \in L_u^c$ iff $w111w \notin L_u$



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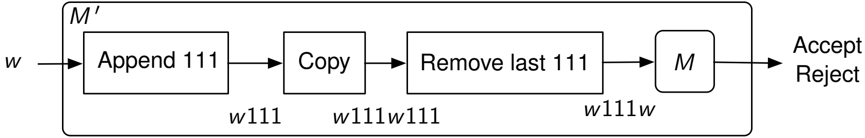

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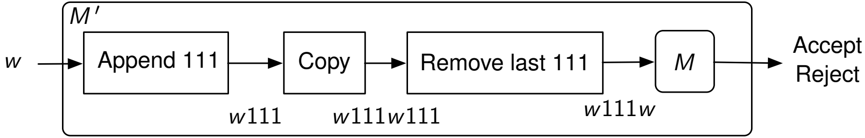

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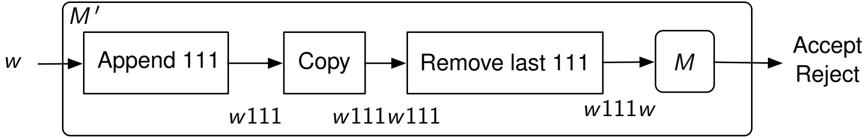
- M' accepts $w \iff w111w \in L_u^c \iff w111w \notin L_u \iff w \in L_d$.
- Then, $L(M')$ is the L_d, for which there is no TM! But M' decides L_d!



Recap

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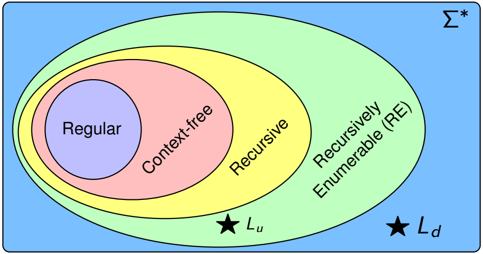
- There exists a bijection $\phi : \Sigma^* \to \mathbb{N}$.
- There exists an injective function $\langle\cdot\rangle$: Set of TMs $\to \Sigma^*$.
- RE languages are countable.



- The diagonalization Language L_d is not recursively enumerable.
- Recursive languages are closed under complementation. (See tutorials for more!)
- The universal language $L_u = \{\langle M\rangle 111w : M \text{ accepts } w\}$ is RE, but not recursive.

Reductions of Problems

What is a Reduction?

- A decision problem P is said to reduce to decision problem Q if **every** instance of P can be transformed to **some** instance of Q and a yes (or no) answer to that instance of Q yields a yes (or no) answer to original instance of P, respectively.
  - We did already make use of reductions in this lecture multiple times!
  - E.g., reduce the problem of deciding L^c to the problem of deciding L: Here the new problem was only a minimal modification, by flipping results (see slide 13).
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Theorem 9.6.1

If a problem P reduces to a problem Q then:

(a) P is undecidable $\Rightarrow$ Q is undecidable.

(b) P is non-RE $\Rightarrow$ Q is non-RE.

Problem Reduction

Proof of Theorem 9.6.1

(a) P **is undecidable** $\Rightarrow$ Q **is undecidable**.
Suppose P is undecidable and Q is decidable. Let TM M_Q decide Q.

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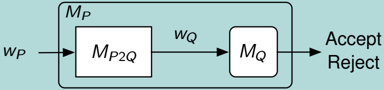

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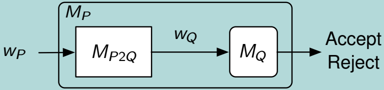

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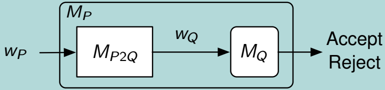

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Suppose P is non-RE and Q is RE. Then there must be a TM M_Q that accepts inputs when they correspond to instances of Q whose answer is yes.

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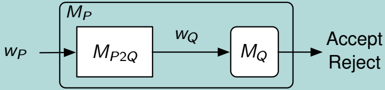

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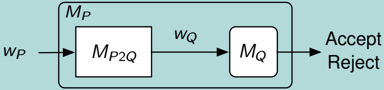

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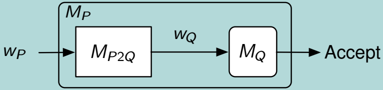

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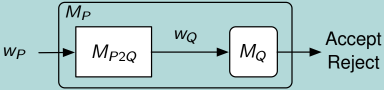

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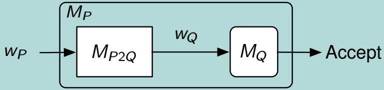

- This is a TM that accepts all instances of P, a contradiction.

Rice's Theorem

Some More Abstract Languages

Language of TMs Accepting Empty and Non-empty Languages

- $L_e = \{\langle M \rangle : L(M) = \emptyset\}$.
- $L_{ne} = \{\langle M \rangle : L(M) \neq \emptyset\}$. (Note: $L_{ne} = L_e^c$ and $L_{ne}^c = L_e$)

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Theorem 9.7.1

L_{ne} is recursively enumerable.

Note that this theorem doesn't say whether it's recursive or not!

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Theorem 9.7.2

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L_{ne} is recursively enumerable.

Theorem 9.7.2

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- Then, execute M on w.
- Accept iff M accepts w.

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Why could we not just iterate over all possible words?

L_{ne} is recursively enumerable.

Theorem 9.7.2

L_{ne} is recursively enumerable.

Proof

- We proved earlier that the set of languages accepted by a non-deterministic TM is the same as the ones accepted by deterministic ones. We hence provide a non-det. TM.
- First, temporarily ignore the input $\langle M \rangle$ and guess an input word w for M.
- Then, execute M on w.
- Accept iff M accepts w.

Why could we not just iterate over all possible words?

Cause we might get stuck in one word! The alternative: "dovetailing", exactly how the proof for compiling away non-determinism worked!

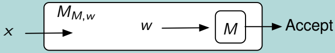
L_{ne} is not recursive

Theorem 9.7.3

L_{ne} is not recursive.

Proof of Theorem 9.7.3

- For every TM M and string w, there is a TM $M_{M,w}$ that ignores its input and runs M on w: $M_{M,w}$ erases its input tape, pastes w, and runs it on M.



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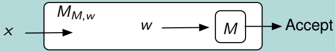

- **Mind-bending step:** There is a TM M_1 that takes $\langle M \rangle 111 w$ and outputs $\langle M_{M,w} \rangle$. Note: M_1 **always** halts (even if M does not halt when input is w!)

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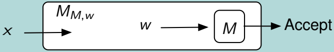

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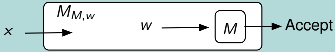

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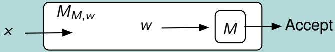

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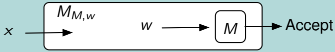

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- Let TM M_3 read $\langle M\rangle 111w$ and operate as M_1, then as M_2, initiated with the output of M_1. Then, M_3 is total and accepts/rejects $\langle M\rangle 111w$ iff M accepts/rejects w.

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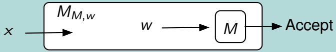

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- We thus decide L_u, which is impossible (it's only recursively enumerable).

Rice's Theorem

Given: alphabet Σ and let $RE = \{L \subseteq \Sigma^* \mid L \text{ is recursively enumerable}\}$.

- A **property** of RE languages is subset $\mathcal{P} \subseteq RE$ of the set of RE languages over Σ. Why do we call sets of languages a property? Think of examples:
  - $\mathcal{P}_1 = \{L \subseteq \Sigma^* : |L| < \infty\}$ (the property is being finite)
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- So, Rice's theorem says something about some (many!) subsets $S \subseteq \{\langle M \rangle : \text{M is a TM}\}$ (So we want to know something about TMs!)

Rice's Theorem (Example 1)

How about the "property" that a TM has 10 states? (Should be decidable!)

- Let $L_{10} = \{\langle M \rangle : \text{M has 10 states}\}$. But we have to be able to write it as: $L_{10} = \{\langle M \rangle : L(M) \in \mathcal{P}\}$ where $\mathcal{P} \subseteq RE$ and not trivial.

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- So how about $\mathcal{P}_{10} = \{L \subseteq \Sigma^* : \text{there is a TM M, s.t. } L = L(M) \text{ and } M \text{ has 10 states}\}$?

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- So how about $\mathcal{P}_{10} = \{L \subseteq \Sigma^* : \text{there is a TM M, s.t. } L = L(M) \text{ and } M \text{ has 10 states}\}$?
- This doesn't work since we can take some M_9 with 9 states (and thus $\langle M_9 \rangle \notin L_{10}$) and add a dummy state, so we have 10 in the resulting TM M_{10}. Now we have:
  - $\langle M_9 \rangle \notin L_{10}$, and $\langle M_{10} \rangle \in L_{10}$, but
  - $L(M_9) = L(M_{10})$, so $L(M_9) \in \mathcal{P}_{10}$ and $L(M_{10}) \in \mathcal{P}_{10}$.
  - Recall $L_{\mathcal{P}} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}$, so $\langle M_9 \rangle \in L_{\mathcal{P}_{10}}$. ↯

$\rightarrow$ So it doesn't work! It's not a property of languages! (So Rice's theorem doesn't apply.)

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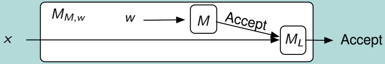
- Thus, $L_{\mathcal{P}_{01}} = \{\langle M \rangle : L(M) \in \mathcal{P}_{01}\}$ is undecidable. In other words: We can't decide whether a given TM accepts a language that contains a 01.

Recap on what that means practically:
For some TMs M, we might be able to correctly answer yes or no – and even terminate!
But we cannot design a single TM D (for decider) that receives as input an arbitrary TM M and we always terminate with the correct yes/no answer!

Rice's Theorem (Proof)

Proof of Theorem 9.7.4

- WLOG, we can assume that $\emptyset \notin \mathcal{P}$. Else consider $\mathcal{P}^c$.
- Since $\mathcal{P}$ is non-trivial, there is a language $L \in \mathcal{P}$ and a TM M_L that accepts L
- Let $M_{M,w}$ be a TM that runs M on w and if M accepts w, then reads its input and operates as M_L.



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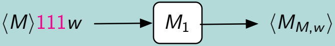

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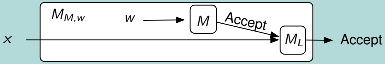

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- If $\mathcal{P}$ were decidable, then there is a TM M_2 such that M_2 accepts $\langle M \rangle$ iff $L(M) \in \mathcal{P}$.
- Then, we can devise a TM M_3 such that it reads $\langle M \rangle 111 w$ operates first as M_1 and then when M_1 has halted, it operates as M_2.
- M_3 accepts/rejects $\langle M \rangle 111 w \iff L(M_{M,w}) \in / \notin \mathcal{P} \iff M$ accepts/rejects w.
- Then, L_u is recursive, a contradiction

Post's Correspondence Problem

PCP: Definition

- Suppose we are given two ordered lists of strings over Σ, say $A = (u_1, \ldots, u_k)$ and $B = (v_1, \ldots, v_k)$. We say (u_i, v_i) to be a **corresponding pair**.
- PCP Problem: Is there a sequence of integers $i_1, \ldots, i_m$ such that:

  $$u_{i_1} \cdots u_{i_m} \\ = v_{i_1} \cdots v_{i_m}?$$

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A PCP example

A	110	0011	0110
B	110110	00	110

- A solution cannot start with $i_1 = 3$.
- A solution can start with $i_1 = 1$, but then $i_2 = 1$, and $i_3 = 1 \ldots$. Consequently, i_1 cannot equal 1.
- A solution does exist: $(i_1, i_2, i_3) = (2, 3, 1)$.
- $(i_1, i_2, i_3, i_4, i_5, i_6) = (2, 3, 1, 2, 3, 1)$ is also a solution.

Modified PCP (MPCP): Definition

- Suppose we are again given two ordered lists of strings over Σ, say $A = (u_1, \ldots, u_k)$ and $B = (v_1, \ldots, v_k)$.
- MPCP Problem: Is there a sequence of integers $i_1, \ldots, i_m$ such that

$$\begin{aligned} & u_1 u_{i_1} \cdots u_{i_m} \\ = \; & v_1 v_{i_1} \cdots v_{i_m} \end{aligned}$$

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$$u_1 u_{i_1} \cdots u_{i_m}$$
$$= v_1 v_{i_1} \cdots v_{i_m}$$
- The previous example does not have a solution when viewed as an MPCP problem.
- So MPCP is indeed a different problem to PCP, but...

Theorem 9.8.1

MPCP reduces to PCP

MPCP: Thoughts/Ideas before constructing a Proof

- So we want to prove that MPCP reduces to PCP.
  (So, PCP is at least as hard as MPCP.)
- More specifically we need to:
  - Turn every MPCP problem into a PCP problem (with preserving solutions).
  - I.e., **how can we enforce PCP to always select the first element first?**

Thus, the problem we need to solve is:

- To make sure that the first string gets selected first, but
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Initial thoughts:

- We add a new start symbol to u_1 and v_1 so that they match.
- But that still doesn't enforce that the "normal PCP" starts with them! ...

Outline of Proof of Theorem 9.8.1

- Given MPCP's lists $A = (u_1, \ldots, u_k)$ and $B = (v_1, \ldots, v_k)$. We now transform this into a PCP problem! Suppose that symbols $\diamond, \triangle$ are not in the strings of A and B.

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- Construct lists $C = (w_0, \dots, w_{k+1})$ and $D = (x_0, \dots, x_{k+1})$ for PCP as follows.
  - For $i = 1, \dots, k$,
    - if $u_i = s_1 \dots s_\ell$, then $w_i = \ s_1 \diamond s_2 \diamond \cdots \diamond s_\ell \diamond$ [$\diamond$ succeeds symbols]
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  - $w_{k+1} = \triangle$ and $x_{k+1} = \diamond\triangle$. [Balances the extra $\diamond$]

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A	B
110	110110
0011	00
0110	110

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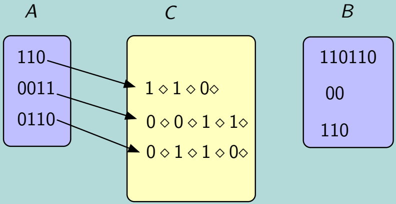

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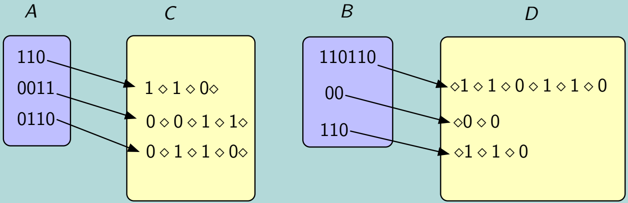

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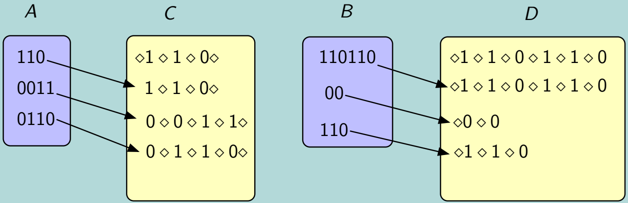

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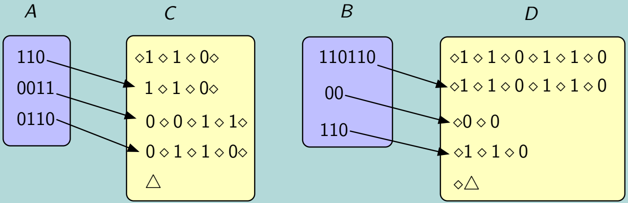

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PCP is undecidable

Theorem 9.8.2

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Outline of Proof of Theorem 9.8.2 (Overview)

We reduce L_u to MPCP (and did already MPCP to PCP).

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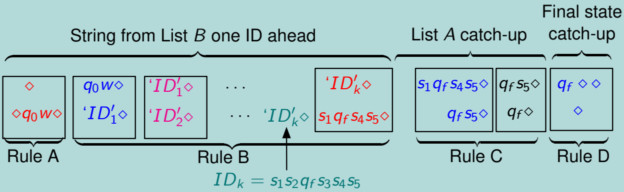
So, the hard work is to solve/model $\langle M \rangle 111w \in L_u$ via MPCP!

PCP is undecidable (More detailed proof at the end)

Outline of Proof of Theorem 9.8.2 (Overview)

Abstract overview of existing pairs in the constructed MPCP:



The rules $A \ldots, D$ are in the appendix.

The overall idea is as follows:

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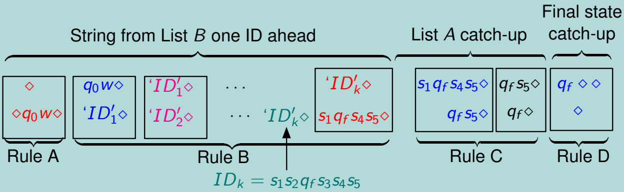

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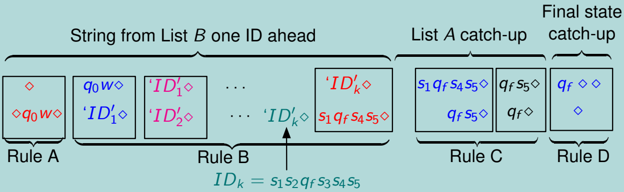

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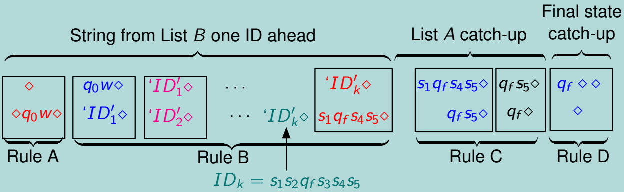

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- We construct a pair for every valid TM transition! (Rule B) In such a pair, the first line/entry is the old configuration and the second the new.
- We have/need a few more rules to make all strings equal and deal with final states. Note how we have to move the first line to get matchings strings. (Rules C, D)

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Proof of Theorem 9.8.2 (Short Example)

Before we look at an example, recap:

- A TM ID looks as: $X_1 \ldots, X_{i-1} q X_i \ldots X_\ell$ where X_i is below the head.

Now, with TM's start state q_0 and initial tape $w = s_1 s_2 s_3$ let:

- Word in line 1: $\diamond$
- Word in line 2: $\diamond q_0 s_1 s_2 s_3 \diamond$

We get this by our first pair, created by Rule A:

- First entry in 1st list: $\diamond$
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What's next? Create the transitions! (Via Rules in B)

- Assume $\delta(q_0, s_1) = (p, t_1, R)$, then $q_0 s_1 s_2 s_3 \underset{M}{\vdash}$
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What's next? The remaining symbols from last configuration are missing...

- We add a pair (s, s) for all $s \in \Gamma$ (Rule I)
- and one pair $(\diamond, \diamond)$ (Rule I)

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We get this by several new pairs, created by Rule I:

- $(s_0, s_0), (s_1, s_1), (s_2, s_2), \dots$ (for all $s \in \Gamma$)
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What's next? We continue! Next transition!

- Assume $\delta(p, s_2) = (r, t_2, L)$, then $t_1 p s_2 s_3 \underset{M}{\vdash}$
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We get this by another pair, created by Rule B:

- Entry in 1st list: $t_1 p s_2$ since $\delta(p, s_2) = (r, t_2, L)$
- Entry in 2nd list: $r t_1 t_2$ and thus $t_1 p s_2 s_3 \underset{M}{\vdash} r t_1 t_2 s_3$

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What's next?

- First, we again add the missing symbols, until
- eventually we find a final state. We have more rules for that (see appendix).

Ambiguity in CFGs/CFLs

- We'll now revisit CFGs and prove that ambiguity in CFGs is undecidable.

Theorem 9.9.1

The problem if a CFG is ambiguous is undecidable.

Outline of Proof of Theorem 9.8.2

- We'll reduce ... which one? (1) PCP to CFG or (2) CFG to PCP?

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Outline of Proof of Theorem 9.8.2

- We'll reduce every instance of a PCP problem to a CFG.
- Given a PCP problem with $A = (w_1, \ldots, w_k)$ and $B = (x_1, \ldots, x_k)$, pick symbols $a_1, \ldots, a_k$ that don't appear in any string in list A or B.
- Now define a grammar G with production rules

$$\begin{aligned}
S &\longrightarrow A \mid B \\
A &\longrightarrow w_1 A a_1 \mid \cdots \mid w_k A a_k \mid w_1 a_1 \mid \cdots \mid w_k a_k \\
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- Now define a grammar G with production rules

$$\begin{aligned} S &\longrightarrow A \mid B \\ A &\longrightarrow w_1 A a_1 \mid \cdots \mid w_k A a_k \mid w_1 a_1 \mid \cdots \mid w_k a_k \\ B &\longrightarrow x_1 B a_1 \mid \cdots \mid x_k B a_k \mid x_1 a_1 \mid \cdots \mid x_k a_k \end{aligned}$$

- If there are two leftmost derivations of a string in $L(G)$, one must use $S \longrightarrow A$ and $S \longrightarrow B$, respectively.
- Every solution to the PCP leads to 2 leftmost derivations of some string in $L(G)$ and vice versa. (Note how the solution indices are encoded in the end of each word.)
- Since PCP is undecidable, the ambiguity of CFGs must be undecidable [Thm 9.6.1]

Overview of (Some) Undecidable Problems Concerning CFGs

- Given a CFG G, is it ambiguous? (We just had that.)

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- Given CFGs G_1 and G_2, is $L(G_1) \subseteq L(G_2)$?

Overview of (Some) Undecidable Problems Concerning CFGs

- Given a CFG G, is it ambiguous? (We just had that.)
- Given CFL L, is it inherently ambiguous?
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- Given CFG G and regular language L, is $L \subseteq L(G)$?
- Given CFG G, is $L(G) = \Sigma^*$?

Appendix: PCP Proof Details

PCP is undecidable

Proof Details of Theorem 9.8.2 (Rule Definitions)

- For the proof we construct an MPCP for each TM M and input w.

Rule A: Construct two lists A and B whose first entries are $\diamond$ and $\diamond q_0 w \diamond$, respectively.

Rule I: Add corresponding pairs (X, X) (for all $X \in \Gamma$) and $(\diamond, \diamond)$

Rule B: Suppose q is not a final state. Then, append to the list the following entries:

List A	List B	
qX	Yp	if $\delta(q, X) = (p, Y, R)$
ZqX	pZY	if $\delta(q, X) = (p, Y, L)$
$q\diamond$	$Yp\diamond$	if $\delta(q, B) = (p, Y, R)$
$Zq\diamond$	$pZY\diamond$	if $\delta(q, B) = (p, Y, L)$

Rule C: For $q \in F$, let (XqY, q), (Xq, q), and (qY, Y) be corresponding pairs for $X, Y \in \Gamma$

Rule D: For $q \in F$ $(q \diamond \diamond, \diamond)$ is a corresponding pair.

PCP is undecidable

Proof Details of Theorem 9.8.2 (Construction/Explanation)

- Suppose there is a solution to the MPCP problem. The solution starts with the first corresponding pair, and the string constructed from List B is already an ID of TM M ahead of the string from List A.
- As we select strings from List A (corresponding to Rule B) to match the last ID, the string from List B adds to its string another valid ID.
- The sequence of IDs constructed are valid sequences of IDs for M starting from q_0w.
- Suppose the last ID constructed in the string constructed from List B corresponds to a final state, then we can gobble up one neighboring symbol at a time using Rule C.
- Once we are done gobbling up all tape symbols, the string from List B is still one final state symbol ahead of List A's string.
- We then use Rule D to match and complete.