COMP3630 / COMP6363

week 7: **Reductions and Complexities**

Not based on the book, but you may read from chapter 10

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The Australian National University

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Content of this Chapter

- Motivation
- Non-Deterministic Turing Machines
- Big-$\mathcal{O}$ Notation
- (Most) Complexity Classes (covered in this course)
- Reductions
- Membership, Hardness, Completeness

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  - Which problems are equally hard? Which ones are harder than others?
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- We measure "performance" in terms of a Turing Machine's:
  - Time requirement (number of operations/transitions)
  - Space requirement (number of cells that can be read/written)

Why should we care for Problem Complexities?

Given a new problem to solve, we:

- ... can use existing solvers instead of designing new ones,
  - $\rightarrow$ Which software do you think is better? The one you design from scratch in a few weeks, or one that entire research communities (few or dozens to thousands of PhD students, post-docs, Professors) created over decades?

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- ... understand the problems we solve much better.
  - → If you know that your (new) problem is equivalent to an existing (established) one, that surely helps... (Imagine, you take a course twice! The second time it's much easier...)

Example (for the Relevance of this)

Boolean Satisfiability (SAT)

- Let ϕ be a boolean formula with n variables, e.g.,:
  $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_4) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_4)$ with $n = 4$.
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  - The first "runtime class" will be called **EXPTIME** (membership)
  - This also takes only polynomial space (if enumeration is implemented in-place), so we are also in **PSPACE**,
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  $\rightarrow$ SAT is in **PSPACE**, **EXPTIME** and in **NP**, but **NP** is lower! (as you will see)
- But is the problem also in **P**? (I.e., can we **P**-decide it without guessing?)

Non-Deterministic Turing Machines

Deterministic Turing Machines, Recap

A Turing Machine has the form $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$, where

- Q: finite set of states
- Σ: finite set of input symbols
- Γ: finite set of tape symbols such that $\Sigma \subseteq \Gamma$
- δ is a (partial) transition function

$$\begin{array}{rcl} \delta \ : \ Q \times \Gamma & \rightharpoonup & Q \times \Gamma \times \{L, R\} \\ (\text{state}, \text{tape symbol}) & \mapsto & (\text{new state}, \text{new tape symbol}, \text{direction}) \end{array}$$

  The direction tells the read/write head which way to go next: Left or Right
- q_0: initial state of the TM
- $B \in \Gamma \setminus B$: is the blank symbol. All but a finite number of tape symbols are Bs
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Fundamental Properties of Non-Det. TMs

Basic Properties/Definitions

- Acceptance:
  - Recap: A det. TM accepts a string w iff $q_0 w \overset{*}{\underset{M}{\vdash}} \alpha p \beta$ for some $p \in F$.
  - For non-det. TMs that's the same, but now there is branching

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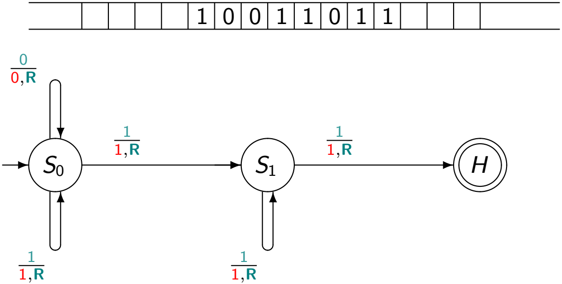
Relationship to Deterministic TMs.

- Non-det. TMs can't do more than deterministic ones. (We proved that!)
- Non-det. TMs could be quicker than deterministic ones. (Unknown!)

Example (for a Non-Det. TM)

Consider the following TM, defined over an initial string over $\Sigma = \{0, 1\}$:



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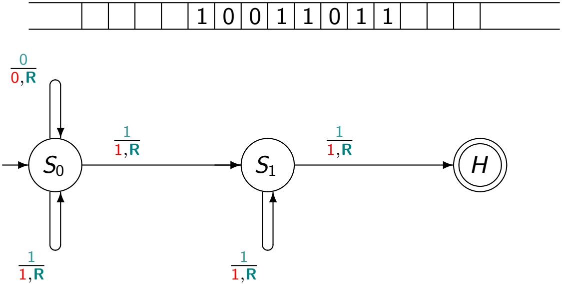

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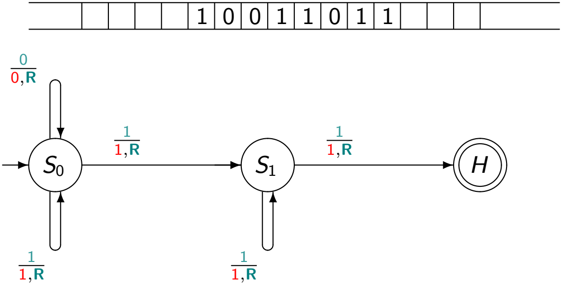

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$$\{w \mid w \text{ contains} \geq 2 \text{ consecutive 1s}\}$$
$$= \{w11w' \mid w, w' \in \Sigma^*\}$$

Big-$\mathcal{O}$ Notation

Introduction

- In the following we will define complexity classes based on whether some TM with specific properties exists.
- Example SAT:
  There is a non-det. TM that runs in "polynomial time" – what does that mean?
- We formalize this using the Big-$\mathcal{O}$ notation.

Poll. Who of you knows the big-$\mathcal{O}$ notation already?

Example

Let's decide $L = \{0^i 1^i \mid i \in \mathbb{N}\}$ by TM M, i.e., check whether an arbitrary input has the form $0^i 1^i$ for some i. M does:

- Scan word w and reject if 10 is found.
- Repeat as long as there are 0s and 1s on the tape:
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   f adheres $f(2k) = f(2(k-1)) + 4k + 1$, which is in $\mathcal{O}(n^2)$.

w	ϵ	01	$0^2 1^2$	$0^3 1^3$	$0^4 1^4$	$0^5 1^5$
$f(\|w\|)$	2	8	19	34	53	76

(exact numbers depend on implementation details)

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2. How much "space" does M need? $\mathcal{O}(n)$

So, in total, M has polynomial time and space restriction!

Time Complexity – Abstraction

Problem. Exact "number of steps function" usually very complicated

- for example, $2n^{17} + 23n^2 - 5$
- and hard to find in the first place (see last slide!).

Solution. Consider approximate number of steps

- focus on asymptotic behavior
- as we are only interested in large problems

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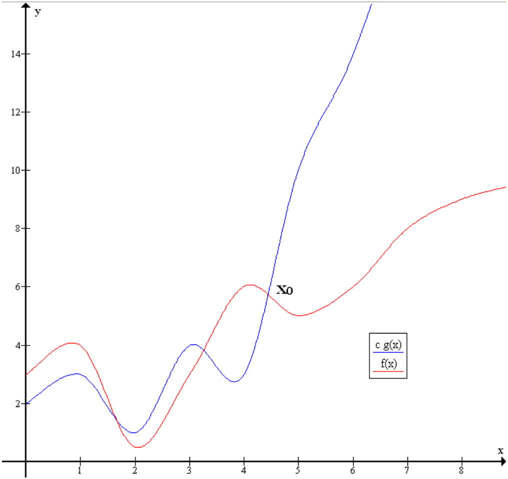
Big-$\mathcal{O}$ Notation. for f and g functions on natural numbers

- $f \in \mathcal{O}(g)$ if $\exists c.\ \exists n_0.\ \forall n \ \geq n_0.\ f(n) \leq c \cdot g(n)$
- "for large n, g is an upper bound to f up to a constant."
- E.g,. $f(n) \in \mathcal{O}(n^{17})$, since $g(n) = n^{17}$ and we can choose $c = 3$ so that we have $3n^{17} \geq f(n)$ for all $n \geq n_0$ (for a suitable n_0)

Graphical Illustration

Recall: $f \in \mathcal{O}(g)$ if $\exists c.\ \exists n_0.\ \forall n \geq n_0.\ f(n) \leq c \cdot g(n)$



Here, $f(x) \in \mathcal{O}(g(x))$ since $g(x)$ is at least as high as $f(x)$ for all $x \geq x_0$

Examples

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- Polynomials: leading exponent dominates
- e.g. "x^n + lower powers of x" $\in \mathcal{O}(x^n)$

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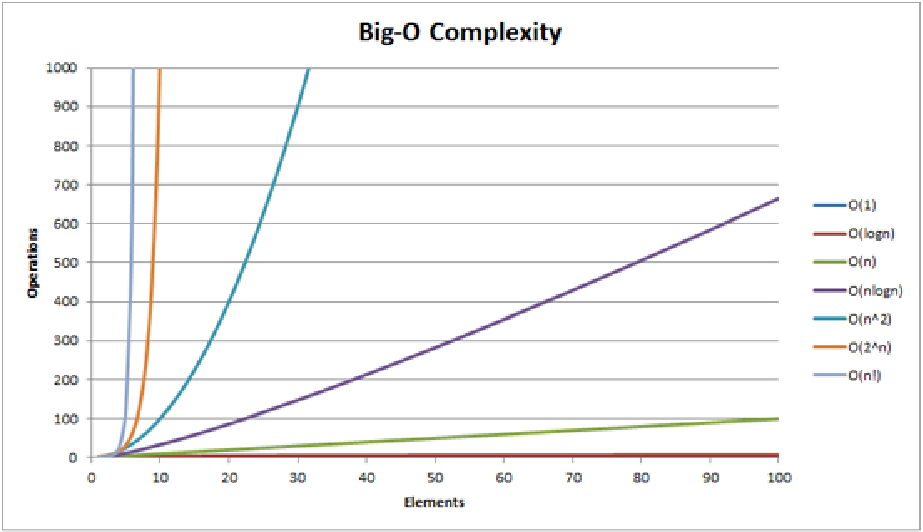
Important Special Cases.

- <u>linear</u>. f is linear if $f \in \mathcal{O}(n)$
- <u>polynomial</u>. f is polynomial if $f \in \mathcal{O}(n^k)$, for some k
- <u>exponential</u>. f is exponential if $f \in \mathcal{O}(2^n)$
- Note: $f \in \mathcal{O}(2^{(n^k)})$ for $k > 1$ is technically not called an exponential, but later used in our definitions of **EXPTIME**.

Important Special Cases, Graphically



(Image copyright Lauren Kroner)

$\log$ in the Big-$\mathcal{O}$ Notation

We may safely omit the base of logarithms in the big-$\mathcal{O}$ notation because

$$\log_a n = \frac{\log_b n}{\log_b a} = \left(\frac{1}{\log_b a}\right) \cdot \log_b n \ .$$

So, two log functions with different bases just differ in a constant multiplier that does not depend on the input. However, we usually don't care for logs anyway, since we will only care whether something is an exponential or polynomial. (And a log doesn't make a difference, there.)

Complexity Classes

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Over the next weeks, we see different problems from these classes and how they relate.

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Answer: <u>Hardness</u> and <u>Completeness!</u> (Later in this section.)

Reductions

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This is the most important (and fun!) part of this week!

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$A \subseteq \Sigma_1^*$ is polynomial time mapping-reducible to $B \subseteq \Sigma_2^*$, written $A \leq_{\mathbf{P}} B$, if a polytime-computable function $f : \Sigma_1^* \to \Sigma_2^*$ exists that is also a reduction (from A to B).

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- $\text{EVEN} := \{n \mid n \bmod 2 = 0\}$, $\text{ODD} := \{n \mid n \bmod 2 = 1\}$
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Heads-up: Be cautious when numbers are involved! They grow exponentially!

Concatenation of **P**-reductions

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Proof.

To decide $w \in A$ first compute $f(w)$ (in **P**) where f is the **P**-reduction from A to B, and then run a **P** decider for B. This is still in **P** because $p_1(p_2(n))$ is a polynomial if $p_1(n)$ and $p_2(n)$ are. □

Examples for what?

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  - Show how the respective problems could be solved (gives class memberships), which is completely irrelevant for (and independent of) reductions!
  - Show the influence of numbers (and hence how tricky they are).

Thus: You should revisit this part after this week, after (**NP**-)completeness and hardness were defined.

Example: Independent Set

The Independent Set Problem:
Assume you want to throw a party. But you know that some of your friends don't get along. You only want to invite people that do get along.

Formalized as graph.

- vertices are your mates
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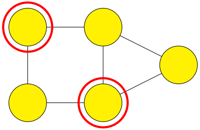

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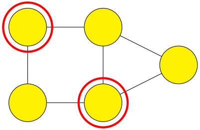
Problem:
Given a graph and $k \geq 0$, is there an independent set, i.e., a subset I of $\geq k$ vertices s.t.:

- no two elements of I are connected with an edge.
- i.e., everybody in I gets along

Formally, $IS := \{(\langle V, E\rangle, k) \mid \langle V, E\rangle$ has an independent set $\geq k,\ k \in \mathbb{N}\}$



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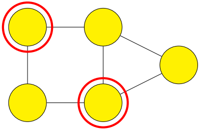

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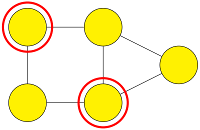

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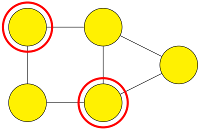

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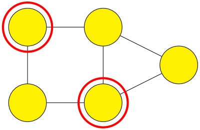

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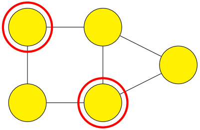

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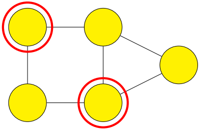

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Using Non-deterministic Turing Machines:

- guess a subset of vertices of size $\geq k$ (with $k \leq |V|$)
- check whether it is an independent set

$\rightarrow$ Proves membership in **NP**

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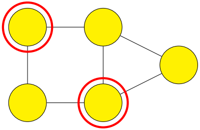

Using Non-deterministic Turing Machines:

- guess a subset of vertices of size $\geq k$ (with $k \leq |V|$)
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Question: Can we do better? Is there a **P** algorithm?

Solving the Independent Set Problem (Irrelevant for Reductions!)

Naive Implementation:

- loop through all subsets of size $\geq k$
  - How many are there? Note that k is an input to the tape: $n_{log(k)} \dots n_1 n_0$
  - So, k grows exponential in $|k| = log(k)$. Always be aware of this when numbers are involved! (So, sometimes this increases the complexity.)
  - There are exponentially many subsets of size $\geq k$. So, why is that not doubly-exponential? $\rightarrow$ Because the nodes are part of the input!
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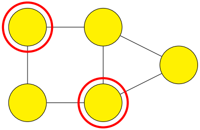

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Answer: We don't know! But "hardness" helps giving a partial answer.

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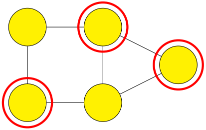
Given a graph $G = \langle V, E \rangle$, a vertex cover is a set C of vertices such that every edge in G has at least one vertex in C.



Example vertex cover: The red-circled vertices. (Since they "cover" all edges.)

Vertex Cover (Decision) Problem.

- Given graph $\langle V, E \rangle$ and $k \geq 0$, is there a vertex cover of size $\leq k$?
- $VC := \{(\langle V, E \rangle, k) \mid \langle V, E \rangle$ has a vertex cover $\leq k,\ k \in \mathbb{N}\ \}$

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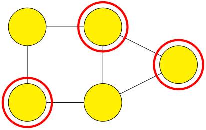

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Naive Algorithm:

- search through all subsets of size $\leq k$ (and $k \leq |V|$), which is exponential
- check whether it's a vertex cover

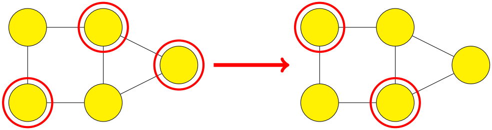
$\rightarrow$ This proves $VC \in$ **EXPTIME**, but we can do better!
(I.e., we could also guess and verify as before, giving $VC \in$ **NP**.)

From Vertex Cover to Independent Set

Reductions. Use solutions of one problem to solve another.

Observation. Let G be a graph with n vertices and $k \geq 0$.

› G has a VC of size $\leq k$ iff G has an IS of size $\geq n - k$



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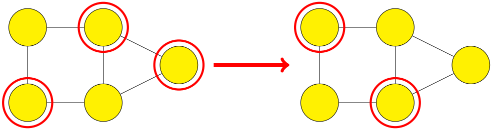

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What's the reduction? Vertex cover to independent set:

- $\langle G, k \rangle \in VC$ iff $r(\langle G, k \rangle) \in IS$, where $= r(\langle G, k \rangle) = \langle G, n - k \rangle$.
- Here, the reduction r only changes the number, but nothing else. But for most reductions, we will have to "translate problems", e.g., when turning a SAT problem into a VC problem! (Or vice versa.)

Important Notes on Reductions

Be aware!

- So far, we only reduced problems, which were "equally hard", they were just "different flavors of the same problem":
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Membership, Hardness, Completeness

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- Thus, a problem having some complexity can be stated in different ways:
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  - Or simply VC $\in$ **NP**
- We'll introduce "hardness" and "completeness" as a boundary for upper and lower bounds of complexity class memberships.

Membership, Hardness, and Completeness

Definition (membership, hardness, completeness)

Let $\mathbf{X}$ *be a complexity class. (Not* $\mathbf{P}$ *in case of hardness.)*

A language B *is* <u>$\mathbf{X}$*-complete*</u> *iff*

1. $B \in \mathbf{X}$ $= \mathbf{X}$ *membership*
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  (Since we decide these other A-problems using our B-problem!)
- Therefore, **X**-complete problems are the hardest ones in **X**.
  For example: we can now say that VC is **NP**-complete, rather than just saying that it's in **NP** – thus indicating that it's a "hard" problem from **NP**, not an easy one that's in **P** (like ODD).

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- Why completeness? Why not just showing hardness?
  - Since the problem could be even harder! (E.g., **PSPACE**-hard, **EXPTIME**-hard, etc., rather than than **NP-complete**.)
  - Each problem class has specific "properties". E.g., "**NP**-complete looks like Logic", "**PSPACE**-complete looks like planning", etc. (But if you only say, e.g., **NP**-hard, it could still be **PSPACE**-complete etc.)

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→ It gives us a convenient procedure to show completeness!

- First, show membership in $\mathbf{X}$. (That's almost always the easy part.)
- Then, show hardness by grabbing an $\mathbf{X}$-complete problem and reduce it to yours!