COMP3630 / COMP6363

week 8: Classes P, NP, and co-NP

This Lecture Covers Chapter 10 and 11 of HMU: *Intractable Problems* and *Additional Classes of Problems*

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The Australian National University

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Content of this Chapter

- Problems in **P**
- Class **NP**
- SAT is **NP**-hard // patient zero!
- Problems that are **NP**-complete
- Class **Co-NP**

Additional Reading: Chapters 10 and 11 of HMU.

Problems in **P**

Context-Free Languages

Theorem w8.1

Every CFL is in **P**.

First, what does that mean? When is a language in **P**?

› $L \in \mathbf{P}$ iff there's a total TM M with $L(M) = L$ that runs in polytime.

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- Note that a TM can represent everything from this course: DFAs, NFAs, grammars, ... Thus, we pick the most suitable representation for which we know a polytime procedure and assume it's given in terms of a TM.

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Proof.

- Let L be context-free.
- We thus know that there exists a CFG G in Chomsky Normal Form (CNF).
- Let G be given. Now run CYK (implemented within our TM) on the input w, taking $\mathcal{O}(|w|^3)$ time. □

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Note that this proof would even be correct if converting a CFG G into CNF would take exponential time! Thus, this theorem does not say that $\{(A, w) \mid w \in L(A)\}$ is polytime decidable for any CFL or automaton A where $L(A)$ is context-free.

More Problems in **P**, cont'd

Theorem w8.2

Deciding $L = \{(G, w) \mid G \text{ is CFG and } w \in L(G)\}$ *is in* **P**.

Proof.

Follows from the last theorem: every CFL is in **P** □

Clear?

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Clear?

It shouldn't, as the proof is wrong ! Why? We never proved that L is context-free! $L(G)$ is context-free for any CFG G, but here your language isn't $L(G)$, but L from above!

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Deciding $L = \{(G, w) \mid G \text{ is CFG and } w \in L(G)\}$ *is in* **P**.

Proof.

- Since G is a CFG, we know that a CFG G' in CNF exists.
- Let M be the TM that implements CYK on G'.
- Run it on w in polytime. Accept or reject accordingly.

□

Correct now?

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No, wrong again ! Why? Because you must provide a TM that receives G and w as input and then run in polytime to make the decision. Here you provide a TM for G', which only works for G/G', but not for arbitrary input words (G, w).

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Proof.

- Since G is a CFG, we know that we can transform G into a CFG G' in CNF in polynomial time.
- Let M be the TM that first does the above normalization and then implements CYK on the resulting CFG G'.
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Correct now?

Still wrong ! Why? Establishing CNF takes exponential time as taught due to elimination of nullable variables, which results in exponentially many new rules.

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- We want to turn CFG G into CFG G' in CNF in polynomial time. The only step that takes exponential time is the elimination of nullable variables.
- So, we first normalize G so that all production rules have fixed size, e.g., 2 (like in step 4). Now, we can eliminate nullable variables in polynomial time.
- Let M be the TM that first does the above normalization and then implements CYK on the resulting CFG G'.
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Yes. :)

More Problems in **P**, cont'd II

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› If n is composite, then it can be written as $n = a \cdot b$ with $1 < a \leq b < n$.

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- Thus, checking all numbers i from 2 to $\lfloor \sqrt{n} \rfloor$ suffices. For each, we test whether $n \bmod i = 0$. If all are "no", return "yes" (prime), otherwise "no". This takes $\mathcal{O}(\sqrt{n}) = \mathcal{O}(n^{\frac{1}{2}})$. We thus get **P** membership. □

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  Since $|n| = \log(n)$, i.e., $n = 2^{|n|}$, we get: $\mathcal{O}(\sqrt{n}) = \mathcal{O}(n^{\frac{1}{2}}) = \mathcal{O}(2^{\frac{|n|}{2}})$.

→ In total, we get $\mathcal{O}(2^{\frac{1}{2}|n|}|n|^2)$. Hence, this algorithm proves **EXPTIME** membership!

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But *PRIME* is indeed in **P**, proved in 2002. (Proof skipped.)

Class **NP**

Conversion of non-det. TM into det. TM

Theorem w8.1

Let $t : \mathbb{N} \to \mathbb{N}$. For every t-time non-deterministic TM M with $L(M) = L$, there is a $2^{\mathcal{O}(t(n))}$-time deterministic TM M' with $L(M') = L$.

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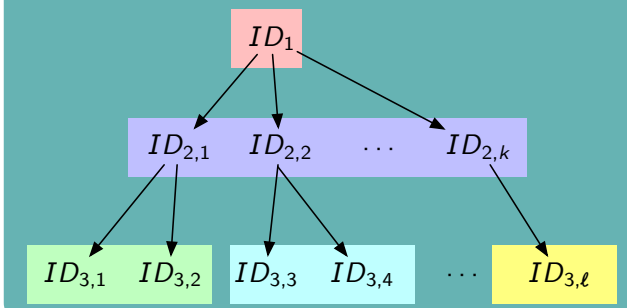
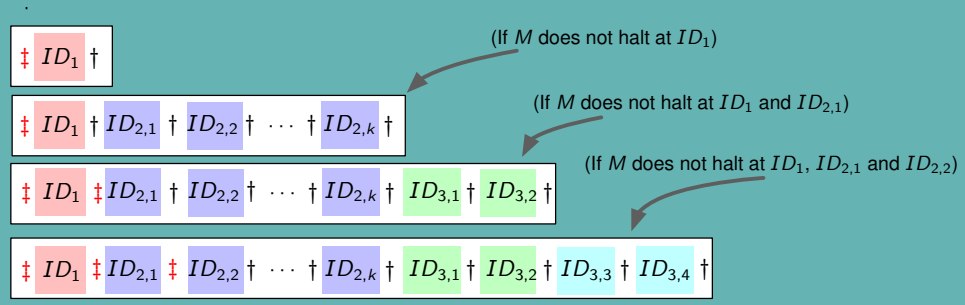
Proof.

By analyzing the time complexity of the construction given to show that every non-deterministic TM has an equivalent deterministic TM. (Cf. week 5)





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› For inputs of length n the computation tree of M has depth at most $t(n)$.

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- M' may have to explore all of them, in a breadth-first fashion. Each exploration may take $\mathcal{O}(t(n))$ steps (from the root to a node).
- So all explorations together may take $\mathcal{O}(t(n)) \cdot \mathcal{O}(b^{t(n)+1}) = 2^{\mathcal{O}(t(n))}$ time.

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How to show **NP** membership

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Thus, to prove **NP** membership of problem P, we had to:

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  - but for some problems, the "property" might be exponentially long! In that case we need a more abstract certificate, as otherwise we could not be in polytime!
- We can conclude that **NP** is not the class of problems that possess a poly-sized "solution" (but a poly-sized "certificate").

Alternative Definition of **NP**

Definition w8.2

A verifier for a language L is a deterministic TM V, such that:
$w \in L$ iff there is a certificate c (a string), such that $(w, c) \in L(V)$

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NP *equals the set of languages that have polynomial-time verifiers with poly-sized certificates (relative to* w, $|c| \leq p(|w|)$ *for some polynomial* p*).*

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$\Rightarrow$: Let $L \in$ **NP**. Then, there exists a poly-time NTM M with $L = L(M)$. Then there must be a sequence of configurations, poly-length-bounded in w, which ends in an accepting state/configuration. Use this sequence as certificate: Design a deterministic verifier V, which tests whether a given sequence of configurations is valid with regard to M and ends in an accepting configuration. Accept/reject accordingly.

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$\Leftarrow$: For L, we know that a (deterministic) verifier V exists, and for each word $w \in L$ a certificate that's poly-length in w. Design a poly-time NTM M with $L = L(M)$ as follows. First, M non-deterministically generates all possible certificates (i.e., each run produces one certificate). Then, we switch into the second phase which implements V, which in turn verifies the certificate and accepts/rejects accordingly. Thus, $L \in$ **NP**. $\square$

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Making our life easier...

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So, we need a very first problem that's shown to be **NP**-hard – from then on we can start reducing!

For this, we will use SAT!

Boolean Formulae

Let $V = \{x, y, \ldots\}$ be a (finite) set of Boolean variables (or propositions).
A CFG to generate well-formed Boolean formulae over V is:

$$\phi \to p \mid \phi \wedge \phi \mid \neg\phi \mid (\phi)$$
$$p \to x \mid y \mid \ldots$$

We use abbreviations such as

$$\phi_1 \vee \phi_2 = \neg(\neg\phi_1 \wedge \neg\phi_2) \qquad \phi_1 \Rightarrow \phi_2 = \neg\phi_1 \vee \phi_2$$
$$\text{FALSE} = (x \wedge \neg x) \qquad \text{TRUE} = \neg\text{FALSE}$$

Semantics of Boolean Formulae

A Boolean formula is either $\top$ (for "true") or $\bot$ (for "false"), possibly depending on the interpretation of its propositions. Let $\mathbb{B} = \{\bot, \top\}$.

Definition w8.1

An interpretation (or assignment or valuation) of V is a function $\pi : V \longrightarrow \mathbb{B}$.
For Boolean formulae ϕ we define that π satisfies ϕ, written $\pi \models \phi$, inductively by:
Base: For each variable $x \in V$, $\pi \models x$ iff $\pi(x) = \top$.
Induction:

- $\pi \models \neg\phi$ iff $\pi \not\models \phi$.
- $\pi \models \phi_1 \wedge \phi_2$ iff both $\pi \models \phi_1$ and $\pi \models \phi_2$.
- $\pi \models (\phi)$ iff $\pi \models \phi$.

ϕ is satisfiable if there exists an interpretation π such that $\pi \models \phi$.

SAT—An **NP**-Complete Problem

Our decision problem/language is:

$$SAT = \{ \langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula } \}$$

Theorem w8.2 (Cook-Levin Theorem – or: Cook's Theorem, 1971/1973)

SAT is **NP***-complete.*

Proof of $SAT \in$ NP.

If $\pi \models \phi$ we use $\langle \pi \rangle$ as certificate. (I.e., guess it and verify.) □

Proof of *SAT* is NP-hard.

The rest of this section. □

Proof of **NP**-Hardness of *SAT*

Let $L \in \mathbf{NP}$. Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a deciding NTM with $L(M) = L$ and let p be a polynomial such that M takes at most $p(|w|)$ steps on any computation for any $w \in \Sigma^*$. (We know that M and p exist due to $L \in \mathbf{NP}$.)

Construct a **P**-reduction from L to *SAT*:

- Input w is turned into a Boolean formula ϕ_w that describes M's possible computations on w.
- M accepts w iff ϕ_w is satisfiable. The satisfying interpretation resolves the nondeterminism in the computation tree to arrive at an accepting branch of the computation tree.

Remains to be done: define ϕ_w.

Proof of **NP**-Hardness of *SAT* cont.

Recall that M accepts w if an $n \leq p(|w|)$ exists and a sequence of configurations $(C_i)_{0 \leq i \leq n}$ (IDs), where

1. $C_0 = q_0 w$,
2. each C_i can yield C_{i+1}, and
3. C_n is an accepting ID.
4. Note that we have at most $n+1$ IDs if the TM can make at most $n \leq p(|w|)$ steps.

ϕ_w

The Boolean formula ϕ_w shall represent all such sequences $(C_i)_{0<i\leq n}$ beginning with $q_0 w$.

$$\phi_w = \phi_{\text{cell}} \wedge \phi_{\text{start}} \wedge \phi_{\text{move}} \wedge \phi_{\text{accept}}$$

The different sub formulae serve the following purposes:

- ϕ_{cell}: Defines all existing "cells", which encode all possible IDs.
- ϕ_{start}: Sets the initial row of these cells: TM's initial ID.
- ϕ_{move}: Enforces legal TM transitions.
- ϕ_{accept}: Enforces ending up in an accepting state.

ϕ_{cell}

. . . describes an n^2 grid using propositions $V = \{ x_{i,k,s} \mid i, k \in \{0, \ldots, n\} \wedge s \in \Sigma_\phi \}$, where $\Sigma_\phi = Q \cup \Gamma$ (recall that $B \in \Gamma$) is the "alphabet of the SAT formula" used to encode the IDs. Also recall that TM IDs contain the non-trivial tape and the state.

First, why is $i, k \in \{0, \ldots, n\}$? Why $s \in \Sigma_\phi$?

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$$\phi_{\text{cell}} = \bigwedge_{0 \leq i,k \leq n} \left(\left(\bigvee_{s \in \Sigma_\phi} x_{i,k,s} \right) \wedge \left(\bigwedge_{s \neq t \in \Sigma_\phi} (\neg x_{i,k,s} \vee \neg x_{i,k,t}) \right) \right)$$

Meaning: "There is exactly one symbol at each cell".

ϕ_{start}

. . . specifies that the first row of the grid contains $q_0 w$ where $w = w_1 \dots w_{|w|}$:

$$\phi_{\text{start}} = x_{0,0,q_0} \wedge \bigwedge_{1 \leq i \leq |w|} x_{0,i,w_i} \wedge \bigwedge_{|w| < i \leq n} x_{0,i,B}$$

So, the first line of our grid contains:

- the q_0 symbol in the first cell,
- followed by the symbols of our initial tape word,
- followed by the blank symbol until the end.

ϕ_{move}

. . . ensures that C_i yields C_{i+1} by describing legal 2×3 windows of cells. We need 3 cells to cover the cell on the left of the state, the state, and on its right (to enable left and right movements of the head).

$$\phi_{\text{move}} = \bigwedge_{0<i,k<n} \bigvee_{\begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline\end{array}\ \text{is legal}} \left(\begin{array}{lll} x_{i,k-1,a_1} & \wedge\, x_{i,k,a_2} & \wedge\, x_{i,k+1,a_3} \wedge \\ x_{i+1,k-1,a_4} & \wedge\, x_{i+1,k,a_5} & \wedge\, x_{i+1,k+1,a_6} \end{array} \right)$$

(Some border cases are not be covered here for simplicity, e.g., i can never be zero.)
What is legal depends on the transition function δ.

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Example: Let the current ID be $w_1 w_2 q w_3 w_4$ (so, we have blanks before and after it). Whether we go to the left or to the right, we only need to change 3 cells!

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Are we still complete?
We can't seem to be able to move to the left of the initial head position!

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Are we still complete?
We can't seem to be able to move to the left of the initial head position!

- Not a problem: We showed equivalence for semi-infinite tapes under polytime.
- We could alternatively have created a grid of size $(2n)^2$, which also goes n to the left.

ϕ_{accept} – and concluding the Proof

. . . states that the accept state is reached:

$$\phi_{\text{accept}} = \bigvee_{0 \leq i,k \leq n, q_F \in F} x_{i,k,q_F}$$

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Concluding the Proof:

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Finally we check that the size of ϕ_w is polynomial in $|w|$ and that ϕ_w is constructable in polynomial time. (Both is true!)

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So, finding a valuation to this formula means deciding $w \in L(M)$ for the arbitrary non-deterministic TM M! So, SAT is **NP**-hard! (It can express every problem in **NP**!)

We have our patient zero now – so, now we can prove **NP**-hardness of other problems by reducing from SAT. (And we build our portfolio...)

NP-complete Problems

CNF-SAT

CNF-SAT is a special case of *SAT*.

$$\textit{CNF-SAT} = \{\ \langle \phi \rangle \mid \phi \text{ is a satisfiable cnf formula } \}$$

where a formula is in cnf (for conjunctive normal form) if it can be generated by a CFG

$$\phi \to (c) \mid (c) \wedge \phi \qquad c \to \ell \mid \ell \vee c$$
$$\ell \to p \mid \neg p \qquad p \to x \mid y \mid \ldots$$

We call cs clauses, ℓs literals, and ps propositions.

Intuitively, a cnf *phi* is simply a conjunction of disjunctions (also called clauses), i.e., $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$, where each ϕ_i is a disjunction

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Example w8.1

$(x \vee z) \wedge (\neg y \vee z)$ is a cnf for the Boolean formula $(x \wedge \neg y) \vee z$.

CNF-SAT is **NP**-Complete

Clearly *CNF-SAT* is in **NP** because we can again guess an assignment and verify it.

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Giving a **P** reduction from *SAT* to *CNF-SAT* is tricky.

A straight-forward translation of Boolean formulae into equivalent cnf may result in an exponential blow-up, meaning that this approach is useless. (Note that such an approach could iterate over all lines in a truth table.)

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A straight-forward translation of Boolean formulae into equivalent cnf may result in an exponential blow-up, meaning that this approach is useless. (Note that such an approach could iterate over all lines in a truth table.)

Instead, we recall a reduction f won't have to preserve satisfaction:

$$\forall \pi \, (\pi \models \phi \quad \Leftrightarrow \quad \pi \models f(\phi))$$

but merely satisfiability

$$\exists \pi \, (\pi \models \phi) \quad \Leftrightarrow \quad \exists \pi \, (\pi \models f(\phi))$$

meaning that we're free to choose different πs for the two sides.

CNF-SAT is **NP**-Hard

The translation from Boolean formulae to cnf proceeds in two steps which are both in **P**.

1. Translate to nnf (negation normal form). (A formula where each negation symbol appears only in front of propositions.)
   This is achieved by pushing all negation symbols down to propositions and eliminating all two consecutive negations. (This is still satisfaction-preserving.)
2. Translate from nnf to cnf. (This merely preserves satisfiability.)

Pushing Down ¬

We use de Morgan's laws and the law of double negation to rewrite left-hand-sides to right-hand-sides:

$$\begin{aligned}
\text{de Morgan on conjunctions:} \quad & \neg(\phi \wedge \psi) \Leftrightarrow \neg(\phi) \vee \neg(\psi) \\
\text{de Morgan on disjunctions:} \quad & \neg(\phi \vee \psi) \Leftrightarrow \neg(\phi) \wedge \neg(\psi) \\
\text{double-negation elimination:} \quad & \neg(\neg(\phi)) \Leftrightarrow \phi
\end{aligned}$$

Example w8.2

$$\neg((\neg(x \vee y)) \wedge (\neg x \vee y)) =$$

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&\quad \Leftrightarrow x \vee y \vee \neg(\neg x \vee y) \\
&\quad \Leftrightarrow x \vee y \vee \neg(\neg x) \wedge \neg y \\
&\quad \Leftrightarrow x \vee y \vee x \wedge \neg y \\
&\quad \Leftrightarrow x \vee y \vee (x \wedge \neg y) \quad \text{This is a disjunction!}
\end{aligned}$$

Pushing Down ¬ cont.

Theorem w8.3

Every Boolean formula ϕ is equivalent to a Boolean formula ψ in nnf. Moreover, $|\psi|$ is linear in $|\phi|$ and ψ can be constructed from ϕ in **P**.

Proof.

By induction on the number n of Boolean operators ($\wedge$, $\vee$, $\neg$) in ϕ we may show that there is an equivalent ψ in nnf with at most $2n-1$ operators. We also have to show that the number of steps is bounded linearly and that each step has polynomial effort. □

nnf $\longrightarrow$ cnf

Theorem w8.4

There is a constant c such that every nnf ϕ has a cnf ψ such that:

1. *ψ consists of at most $|\phi|$ clauses.*
2. *ψ is constructable from ϕ in time at most $c|\phi|^2$.*
3. *$\pi \models \phi$ iff there exists an extension π' of π satisfying $\pi' \models \psi$, for all interpretations π of the propositions in ϕ*

Thus, we can turn any nnf ψ into cnf in polynomial time.

Proof.

By induction on $|\phi|$. □

nnf $\longrightarrow$ cnf cont.

The transformation is done by the Tseytin transformation (from 1968).
Example taken from Wikipedia.

Example w8.5

Let $\phi = ((p \vee q) \wedge r) \rightarrow (\neg s)$. We introduce new auxiliary variables for all subformulae:

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$$x_1 \leftrightarrow \neg s \qquad x_2 \leftrightarrow p \vee q \qquad x_3 \leftrightarrow x_2 \wedge r \qquad x_4 \leftrightarrow x_3 \rightarrow x_1$$

Now we can express ϕ as the following:

nnf $\longrightarrow$ cnf cont.

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$$\psi = x_4 \wedge (x_4 \leftrightarrow x_3 \rightarrow x_1) \wedge (x_3 \leftrightarrow x_2 \wedge r) \wedge (x_2 \leftrightarrow p \vee q) \wedge (x_1 \leftrightarrow \neg s)$$

Each conjunct can be turned (in polytime) into a cnf, e.g.,

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Each conjunct can be turned (in polytime) into a cnf, e.g.,

$$\begin{aligned} x_2 \leftrightarrow (p \vee q) &\equiv x_2 \rightarrow (p \vee q) \wedge ((p \vee q) \rightarrow x_2) \\ &\equiv (\neg x_2 \vee p \vee q) \wedge (\neg(p \vee q) \vee x_2) \\ &\equiv (\neg x_2 \vee p \vee q) \wedge ((\neg p \wedge \neg q) \vee x_2) \\ &\equiv (\neg x_2 \vee p \vee q) \wedge (\neg p \vee x_2) \wedge (\neg q \vee x_2) \end{aligned}$$

Conclusion

We proved that *CNF-SAT* is **NP**-hard!

We reduced: $SAT \leq_{\mathbf{P}} nnf \leq_{\mathbf{P}} CNF\text{-}SAT$

Since *CNF-SAT* is clearly in **NP** as well, it's **NP**-complete.

3-SAT

3-SAT is a special case of *CNF-SAT*.

$$\textit{3-SAT} = \{\ \langle \phi \rangle \mid \phi \text{ is a satisfiable 3cnf formula } \}$$

where a formula is in 3cnf (for 3 literal cnf) if it can be generated by a CFG

$$\phi \to (c) \mid (c) \wedge \phi \qquad c \to \ell \vee \ell \vee \ell$$
$$\ell \to p \mid \neg p \qquad p \to x \mid y \mid \ldots$$

Intuitively, a 3cnf is simply a conjunction of disjunctions of size exactly 3.

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3-SAT is a special case of *CNF-SAT*.

$$\textit{3-SAT} = \{\ \langle \phi \rangle \mid \phi \text{ is a satisfiable 3cnf formula } \}$$

where a formula is in 3cnf (for 3 literal cnf) if it can be generated by a CFG

$$\phi \to (c) \mid (c) \wedge \phi \qquad c \to \ell \vee \ell \vee \ell$$
$$\ell \to p \mid \neg p \qquad p \to x \mid y \mid \ldots$$

Intuitively, a 3cnf is simply a conjunction of disjunctions of size exactly 3.

Example w8.6

$(x \vee y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (x \vee \neg y \vee \neg z)$ is a 3cnf for the Boolean formula x. (You can verify this by applying simplification rules or constructing a truth table.)

3-SAT is **NP**-Complete

Theorem w8.7

3-SAT is **NP***-Complete*

Proof.

As before: Guess a valuation and verify.

We **P**-reduce from *CNF-SAT* to *3-SAT*, by translating arbitrary clauses into clauses with exactly three literals. (We do this on the next slides.) □

Proof: *3-SAT* is **NP**-hard

How to transform a cnf $\phi = \bigwedge_{i=1}^{n} c_i$ into an equisatisfiable 3cnf?

We transform each clause $c_i = \bigvee_{j=1}^{k_i} \ell_{i,j}$ depending on the number k_i of literals in it.
E.g., $c_2 = l_{2,1} \vee l_{2,2} \vee l_{2,3} \vee l_{2,4}$ with $k_2 = 4$. We omit subscript i! $c = l_1 \vee l_2 \vee l_3 \vee l_4$.

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Case $k = 1$ (ℓ_1) is replaced by

$$(\ell_1 \vee u \vee v) \wedge (\ell_1 \vee u \vee \neg v) \wedge (\ell_1 \vee \neg u \vee v) \wedge (\ell_1 \vee \neg u \vee \neg v)$$

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Case $k = 3$ is 3cnf already.

Case $k > 3$ $(\bigvee_{j=1}^{k} \ell_j)$. On the next slide!

Proof: *3-SAT* is **NP**-hard

Case $k > 3$, $(\bigvee_{j=1}^{k} \ell_j)$ is replaced by

$$(\ell_1 \vee \ell_2 \vee u_1) \wedge \bigwedge_{j=1}^{k-4} (\ell_{j+2} \vee \neg u_j \vee u_{j+1}) \wedge (\neg u_{k-3} \vee \ell_{k-1} \vee \ell_k)$$

for some $k - 3$ fresh propositions $u_1, \ldots, u_{k-3}$.

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for some $k-3$ fresh propositions $u_1, \ldots, u_{k-3}$.

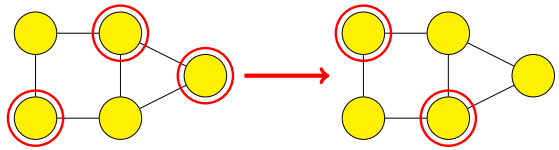
Take $l_1 \vee l_2 \vee l_3 \vee l_4 \vee l_5 \vee l_6 \vee l_7$. So $k = 7$ and $k - 3 = 4$. We can write this as:

$$\begin{aligned}
&(l_1 \vee l_2 \vee u_1) \wedge \\
&\quad (l_3 \vee \neg u_1 \vee u_2) \wedge \\
&\quad (l_4 \vee \neg u_2 \vee u_3) \wedge \\
&\quad (l_5 \vee \neg u_3 \vee u_4) \wedge \\
&(\neg u_4 \vee l_6 \vee l_7)
\end{aligned}$$

You can see that you can always pick the new propositions in a way to make all disjuncts true, no matter which literal is supposed to get true. E.g., if l_4 is true, we set u_1, u_2, u_4 true. Likewise, they don't help us making the formula true unless at least one of the l_i are true. (Check what happens if all l_i are false.)

Vertex Cover vs. Independent Set

Recap from week 7 (more formally)



Theorem w8.8

A graph G with $|V| = n$ vertices has a vertex cover C of size $|C| = k$ iff it has an independent set of size $n - k$. (Both problems are polytime-reducible to each other.)

Proof.

Let G be a graph with n nodes. Let $0 \leq k \leq n$.

Claim: C is a vertex cover of G iff $V \setminus C$ is an independent set.

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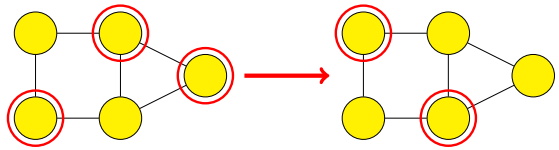

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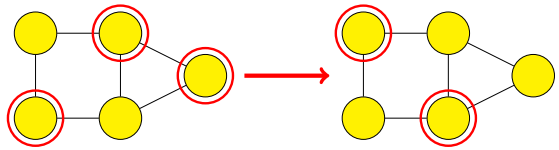

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"$\not\Rightarrow$" C is not a vertex cover of G. Thus there is an edge (v_1, v_2), such that neither of these nodes are in C, $v_1, v_2 \notin C$. But then $v_1, v_2 \in V \setminus C$. Therefore $V \setminus C$ is not an independent set. □

On the **NP**-completeness of these Problems

So far we've shown that both problems are equivalent, so how hard are they?

in **NP** Both problems are in **NP**: We can guess the respective set of nodes and check the required property. The number of guessed nodes is polytime-bounded in the input, and the verification can also be done in polytime.

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NP hard Since we saw that both problems can be turned into each other (even trivially), we can choose for which we show hardness! Completeness then follows for both.

We show hardness for Vertex Cover.

Theorem w8.9

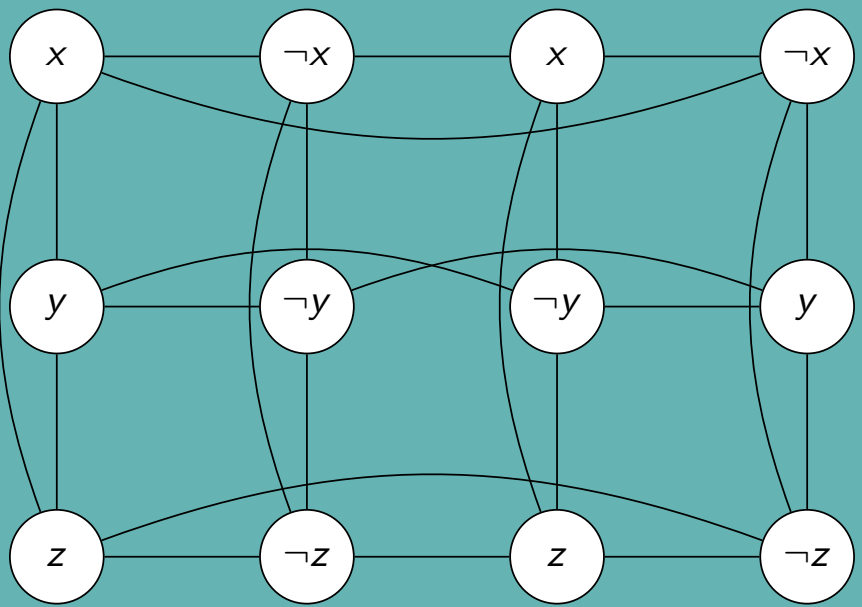
Vertex Cover is **NP***-hard.*

NP-hardness of Vertex Cover

Proof.

We reduce 3-SAT to Vertex Cover.

Let $\phi = (x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (\neg x \vee y \vee \neg z)$.



› We have one column per clause.

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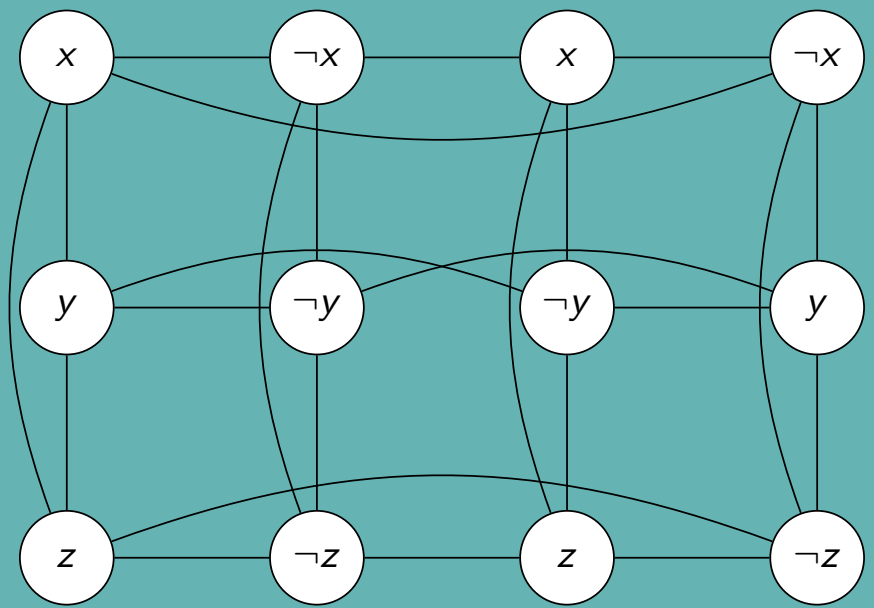

- We have one column per clause.
- Vertically, we connect all nodes within one column.
- Horizontally, we connect all contradictory nodes.

For example, $\pi(x) = \top, \pi(y) = \top, \pi(z) = \bot$ makes the formula true.

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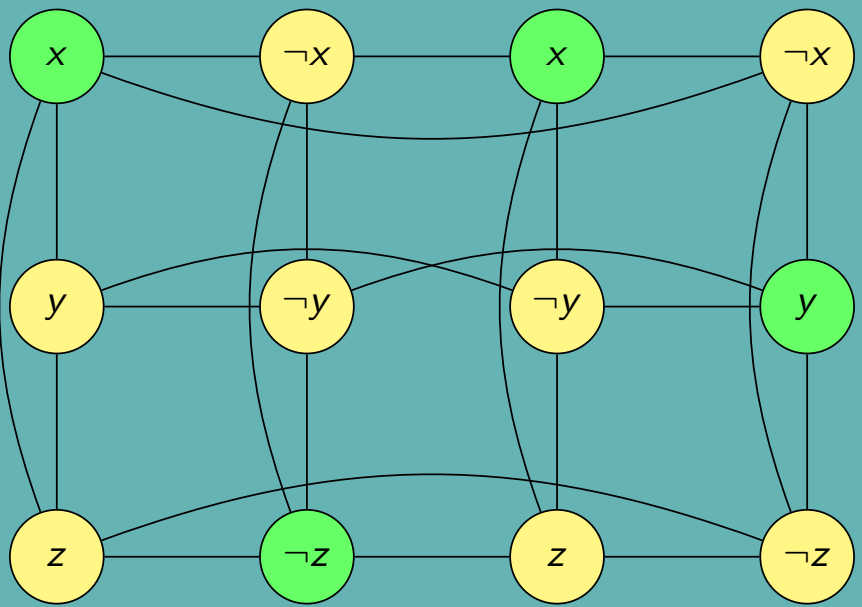


- We have one column per clause.
- Vertically, we connect all nodes within one column.
- Horizontally, we connect all contradictory nodes.
- **We claim:** ϕ is satisfiable iff G has a vertex cover of size $k = 2n$, where $n = 4$ is the number of clauses (select two from each column). The non-selected ones encode the literal that makes the respective clause true.

For example, $\pi(x) = \top, \pi(y) = \top, \pi(z) = \bot$ makes the formula true.

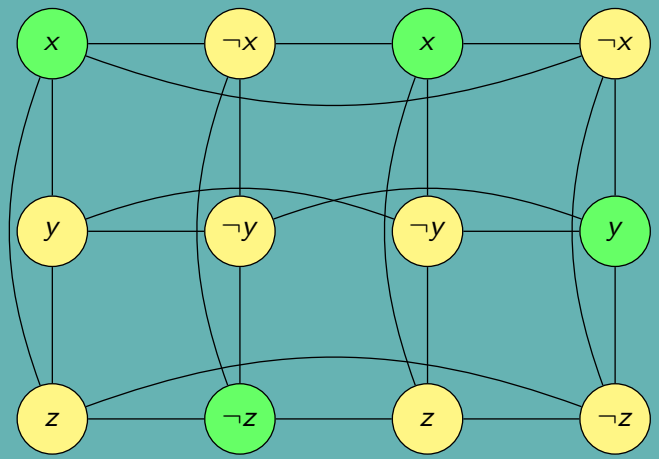
Now we still need to show this claim!

NP-hardness of Vertex Cover (cont'd)

Proof. (Reduction, "⇒").

Recall: ϕ is satisfiable iff G has a vertex cover of size $k = 2n$

Let π make ϕ true, $\pi \models \phi$. Then for all clauses $i = 1, ..., n$, there is (at least) one literal l_i of ϕ_i, s.t. $\phi \models l_i$. E.g., let $l_1, \ldots, l_4$ be the green nodes (non-selected by cover).



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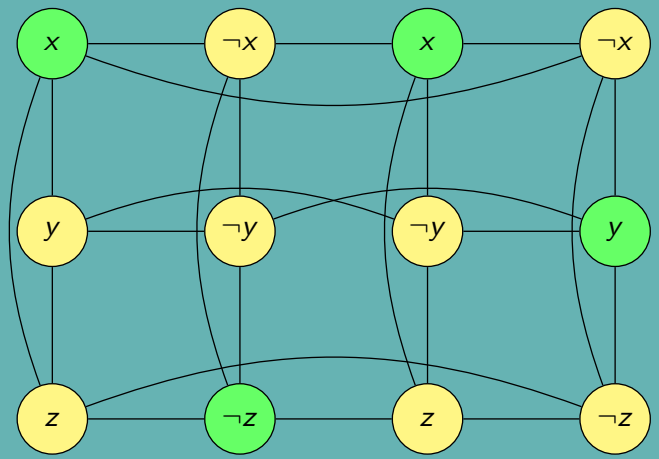
Now define the complement of these nodes as the vertex cover C (the yellow nodes) and show desired properties, i.e., that for each edge (v_1, v_2), at least v_1 or v_2 is in C.

vertical

horizontal



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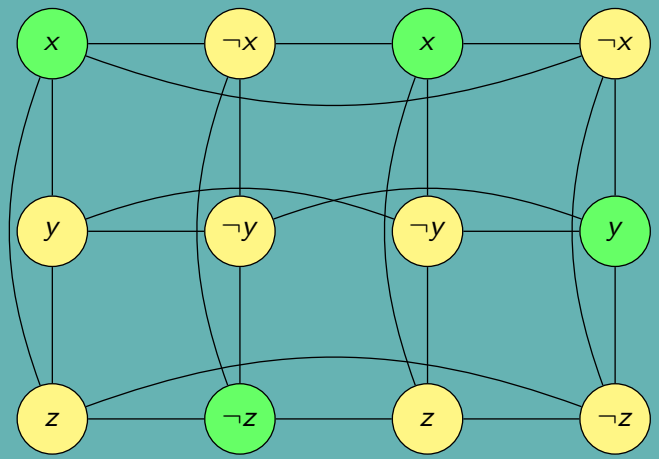

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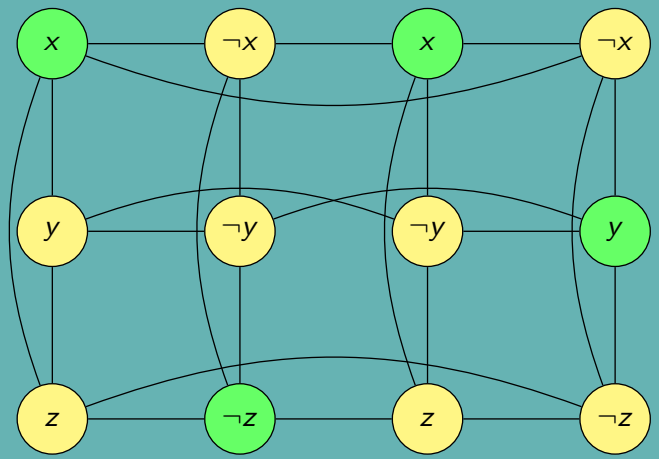

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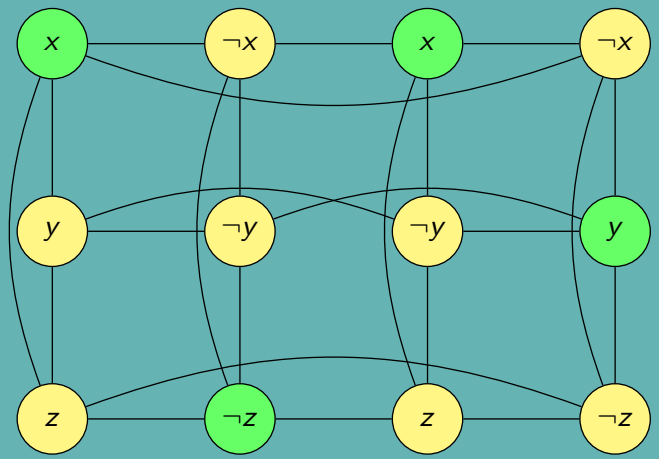

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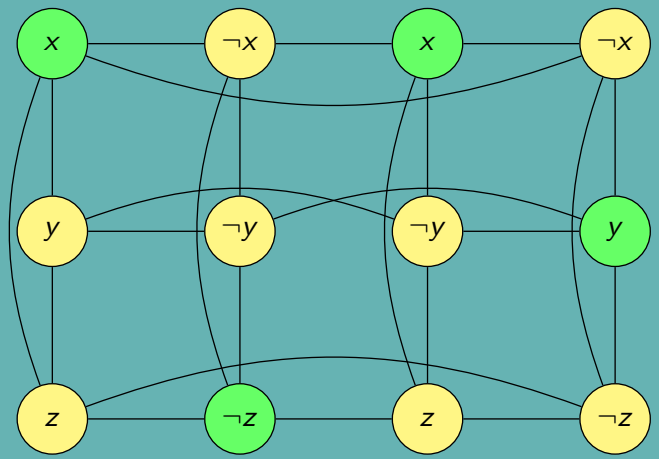

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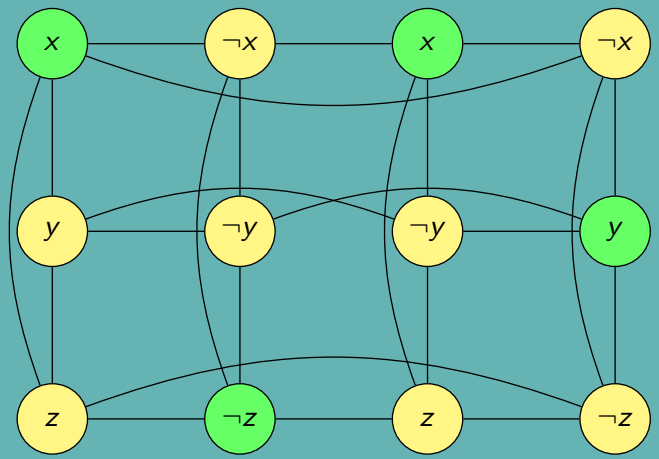

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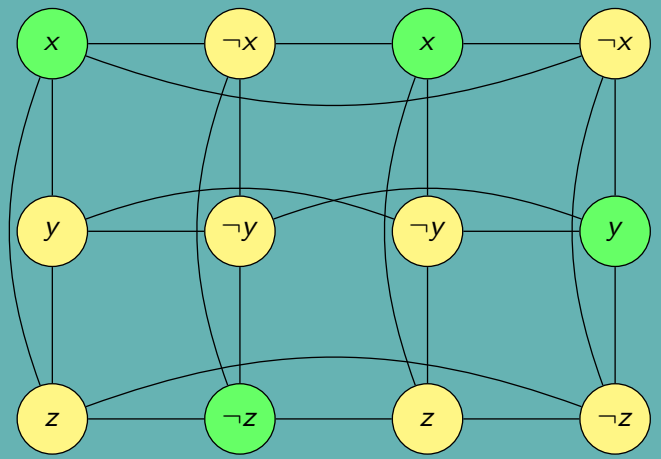
A. Then another "column" doesn't work.



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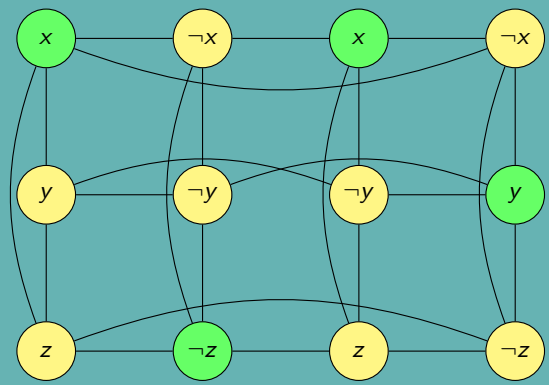

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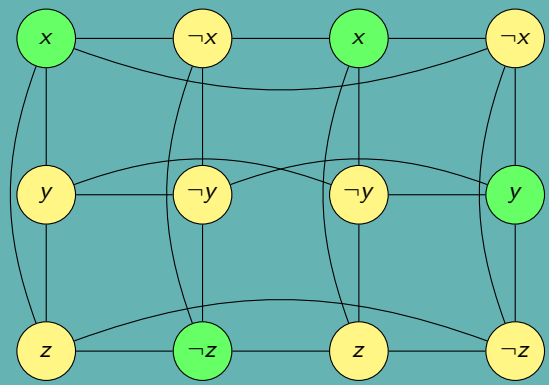

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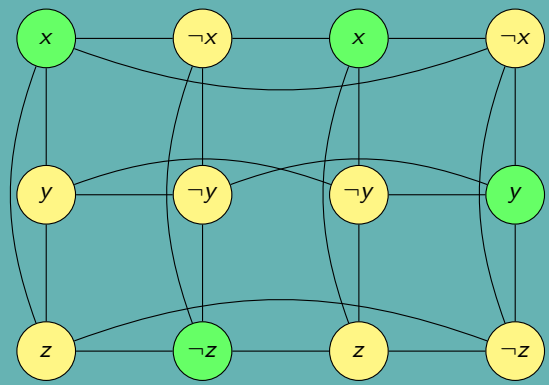

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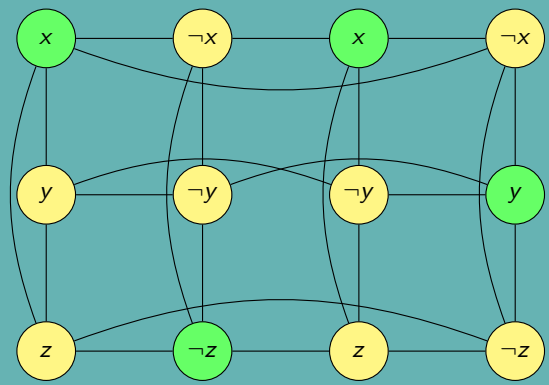

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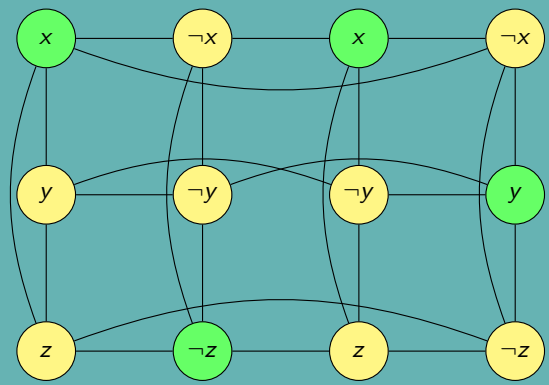

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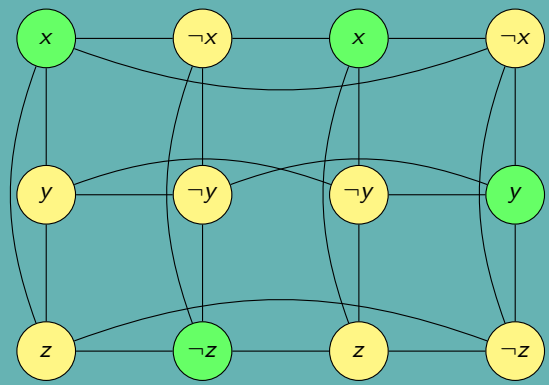

So what could still go wrong? We need consistent assignments!

- I.e., don't make some literal l_i true and false, $\pi(l_i) = \pi(\neg l_i) = \top$.
- Can't happen! They all share a (horizontal) edge, so using both in the assignment (green) would exclude them for the vertex cover – leaving a non-covered edge.

□

Summary of **NP**-complete Problems

We proved the **NP**-completeness of:

- SAT, CNF-SAT, 3-SAT
- Set Cover, Independent Set (from Karp's 21)

In the tutorials, you will see:

- k-SAT
- Partition-Clique
- Graph Colouring, 3-Colouring (from Karp's 21)
- Integer Linear Programs (ILPs)

Other important **NP**-complete problems you should know:

- Karp's 21 **NP**-complete problems (from his 1972 paper)
- Clique (from Karp's 21)
- Hamiltonian Path (a variant from one of Kar's 21)
- Travelling Salesman Problem (from 1930 or earlier)

co-NP and **co-P**

The class co-NP

Definition w8.1

A problem is in **co-X** if and only if its complement is in **X**.

Phrased more formally, applied to $\mathbf{X} = \mathbf{NP}$: Let $\bar{L} = \Sigma^* \setminus L$

$$\textbf{co-NTIME}(t(n)) = \{L \mid \bar{L} \in \textbf{NTIME}(t(n))\} \qquad \textbf{co-NP} = \bigcup_{k \in \mathbb{N}} \textbf{co-NTIME}(n^k)$$

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- No matter whether $w \in L$ or $w \notin L$, the decision can be made in (non-deterministic) polytime, both for $L \in \mathbf{NP}$ and/or $L \in \mathbf{co\text{-}NP}$.
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- For **NP** versus **co-NP** problems, we get different properties:
  - For $L \in \mathbf{NP}$, we can provide a certificate for yes-instances (i.e., $w \in L$)
    - $w \in L$: The non-det. TM (for L) accepts w on at least one path.
    - $w \notin L$: The non-det. TM (for L) rejects w on all paths.
  - For $L \in \mathbf{co\text{-}NP}$, we can provide a certificate for no-instances (i.e., $w \notin L$)
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Properties of **co-NP** and **co-P**

Note that it's still not known whether $\mathbf{NP} \overset{?}{=} \mathbf{co\text{-}NP}$.

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- What if $\mathbf{NP} \neq \mathbf{co\text{-}NP}$? Then, there are problems that guarantee poly-size certificates for only one answer.

co-NP-Hardness and -Completeness vs. NP ...

Remark: Hardness and Completeness for **Co-NP** are defined as for any other class:

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A problem B is **co-NP**-hard if every $A \in$ **co-NP** is **P**-reducible to B.
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That depends on our technology and assumptions!

- Solvers are usually optimized to find solutions, not to disprove their existence.
- Thus, if you expect your problem to have the respective property, **NP** works better, otherwise **co-NP**. Thus,
  - Proving $w \in L$ for L **NP**-complete is usually easy, proving $w \notin L$ is hard.
  - Proving $w \notin L$ for L **co-NP**-complete is usually easy, proving $w \in L$ is hard.

  Because solvers find certificates!

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Let's go through SAT:

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Theorem w8.4

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Proof.

- Hardness: Covered in the tutorial.
- Membership:
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  Correct?

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  Now! :) We return false if we can guess an assignment that makes it false. (And true otherwise.)

□

Keep in mind: For **co-NP** membership we need to show that the complement is in **NP**. The complement of a tautology is that that there exists an assignment that makes it true. That's what checked in **NP**.

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If TAUT is in **P**, *then every problem in* **NP** *is in* **P**.

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We show that we could solve any *SAT* problem in **P** if *TAUT* is in **P**. (*SAT* is **NP**-hard!)

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Questions

- Have we shown that *TAUT* is **NP**-hard?
- **No!** This was not a polytime reduction from *SAT* to *TAUT*. Why?
- Because we flipped the result! We don't implement $w \in SAT$ iff $f(w) \in TAUT$.