COMP3630 / COMP6363

week 9: PSPACE to (N)EXPTIME

This Lecture Covers Chapter 11 of HMU: Additional Classes of Problems

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Content of this Chapter

- A **PSPACE**-complete problem: QBF
- **PSPACE** vs. **NPSPACE** (Savitch's Theorem)
- **PSPACE** vs. **co-PSPACE**
- **P** vs. **PSPACE** vs. **EXPTIME**
- Overview and Outlook of other classes

Additional Reading: Chapter 11 of HMU.

A **PSPACE**-complete Problem: QBFs

Quantified Boolean Formulae (QBFs)

Definition w9.1

If V is a set of variables, then the set of quantified boolean formulae over V is given by:

- Every variable $v \in V$ is a QBF, and so are $\top$ and $\bot$
- If ϕ, ψ are QBF, then so are $\phi \wedge \psi$ and $\phi \vee \psi$
- If ϕ is a QBF, then so is $\neg\phi$.
- If ϕ is a QBF and $v \in V$ is a variable, then $(\exists v)\phi$ and $(\forall v)\psi$ are QBF.

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Definition w9.2

In a QBF ϕ, a variable v is bound if it is in the scope of a quantifier $\forall v$ or $\exists v$. The variable v is free otherwise.

If $x \in \{\top, \bot\}$ is a truth value, then $\phi[x/v]$ is the result of replacing all free occurrences of v with x (think of $/$ as $=:$, i.e., $x =: v$).

Example

$$(\forall x)\,(\exists y)\,(\,((\exists x)\,(x \vee y)) \wedge \neg(x \wedge y)\,)$$

- Usually, one writes these formulae without the parentheses pairs around the quantified variables, e.g, $\forall x \phi$ instead of $(\forall x)\phi$.
- Note how inner quantifiers have precedence over outer ones.
- Also, this formula does not have free variables, i.e., all are bound.

The formula above has the following policy as solution:

x	y	z
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$

$$x = \bot: \quad (\top \vee \top) \wedge \neg(\bot \wedge \top) = \top \wedge \neg\bot = \top$$
$$x = \top: \quad (\top \vee \bot) \wedge \neg(\top \wedge \bot) = \top \wedge \neg\bot = \top$$

- We have two lines because we need to provide an assignment for each x
- Had we more universally quantified variables, we had more "branching"

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$$(\forall x)\,(\exists y)\,(\,((\exists x)\,(x \vee y)) \wedge \neg(x \wedge y)\,) = (\forall x)\,(\exists y)\,(\,((\exists z)\,(z \vee y)) \wedge \neg(x \wedge y)\,)$$

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Evaluation of QBFs

Observation.

A QBF ϕ without free variables can be evaluated to a truth value:

- $\text{evalQBF}(\forall v \phi) = \phi[\top/v] \wedge \phi[\bot/v]$
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QBFs versus boolean formulae.

A boolean formula ϕ with variables $v_1, \ldots, v_n$ is:

- satisfiable if $\exists v_1 \exists v_2 \ldots \exists v_n \phi$ evaluates to true.
- a tautology if $\forall v_1 \forall v_2 \ldots \forall v_n \phi$ evaluates to true.

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Definition w9.3

The QBF problem is the problem of determining whether a given quantified boolean formula without free variables evaluates to true:

$$\text{QBF} = \{\langle \phi \rangle \mid \phi \text{ a true QBF without free variables}\}$$

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- Which runtime does this approach have? Shows **EXPTIME** membership due to doubling the formula with each substitution.

Q. Can we do better?

A. Interestingly, we don't need to reduce the runtime below **EXPTIME**, so it's fine that even a "better result" still requires **EXPTIME**! But we need to reduce the space requirements to **PSPACE** to get to a lower class. (So, note that the procedure above also incurs **EXPSPACE**.)

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Main Idea.

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Analysis.

Given QBF ϕ of size n:

- at most n recursive calls active
- each call stores a partially evaluated QBF of size n
- total space requirement $\mathcal{O}(n^2)$

This shows **PSPACE** membership of QBF.

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- Construct QBF ϕ: "there is a sequence of $c^{p(n)}$ IDs that accepts w".
- Use "recursive doubling" to perform this reduction in polytime.

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- For SAT, we used boolean formulae to represent poly many poly-bounded IDs.
- Now, we use QBFs to represent all runs of poly-bounded IDs.

Hardness Proof, Overview

Variables.

- We use two sets of variables, $x_{j,s}$ and $y_{j,s}$. Need $\mathcal{O}(p(n))$ variables to represent an ID:
- variables $x_{j,s}/y_{j,s} = \top$ iff the j-th symbol of the resp. ID, $0 \leq j \leq p(n)$, is s.

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Structure of the QBF.

$$\phi = (\exists X)(\exists Y)(S \wedge N \wedge A \wedge U)$$

- We use X as the tuple of all x-variables, and Y as the tuple of all y-variables. They will be used to encode the current and successor configuration.

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- **N**: transition from $X \approx ID_0$ to some $Y \approx ID_a$ in $\leq c^{p(n)}$ steps (see next slide).

Recursive Doubling

- $N = N(ID_0, ID_a)$: have sequence of length $\leq c^{p(n)}$ from (formula satisfying) ID_0 to (formula satisfying) ID_a. (I.e., from the start ID to some accepting ID)

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- Detour: $N_i(X, Y) = X \vdash^* Y$ in $\leq 2^i$ steps:
  - Could also say $(\exists K)(N_{i-1}(X, K) \wedge N_{i-1}(K, Y))$
  - this would write out N_{i-1} twice, doubling formula size at each step
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By construction, $\phi = \top$ iff M accepts w.

PSPACE vs. **NPSPACE**
(Savitch's Theorem)

Savitch's Theorem: **PSPACE** = **NPSPACE**

Note

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PSPACE = **NPSPACE** *Savitch's Theorem, 1970*

Proof.

› Let $L \in$ **NPSPACE** and M be non-det. TM, polyspace-bounded by $p(n)$ deciding L.

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Savitch's Theorem: Recursive Doubling

Goal. Implement $P(I, J, m) = I \vdash^* J$ in deterministic polyspace

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P(I, J, m): for all IDs K with length <= p(n) + 1 do {
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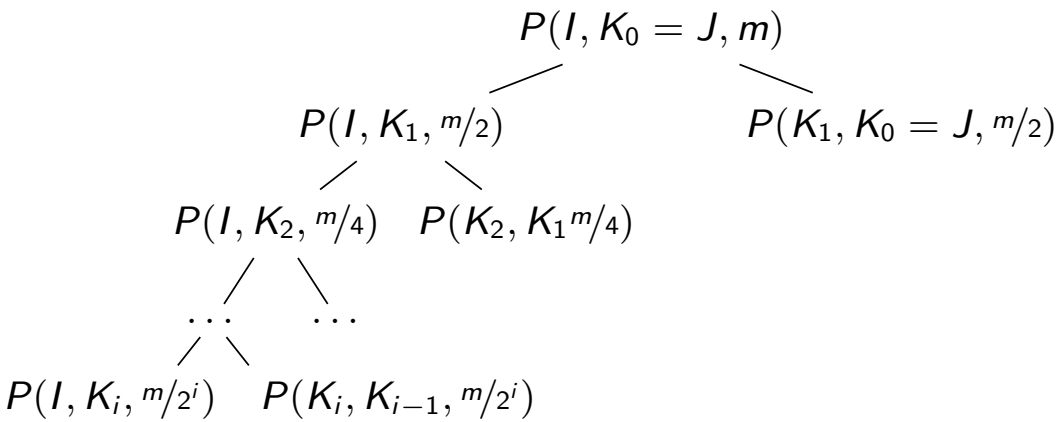

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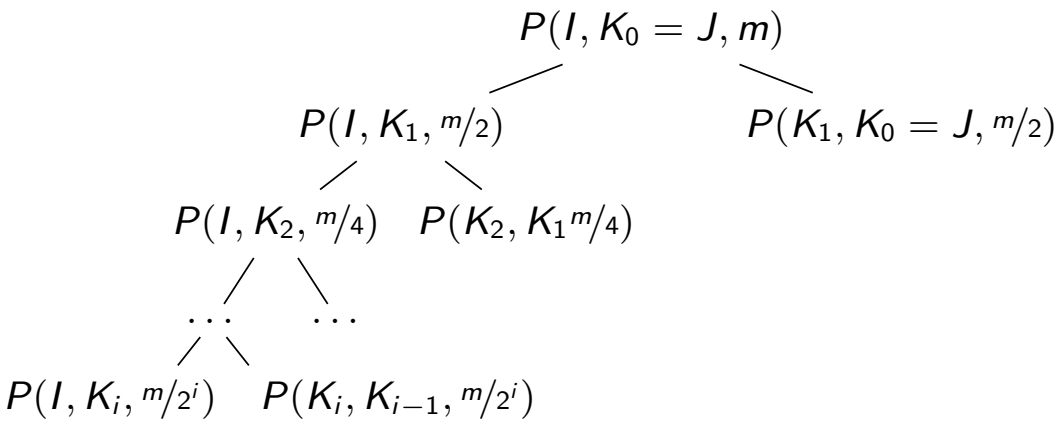

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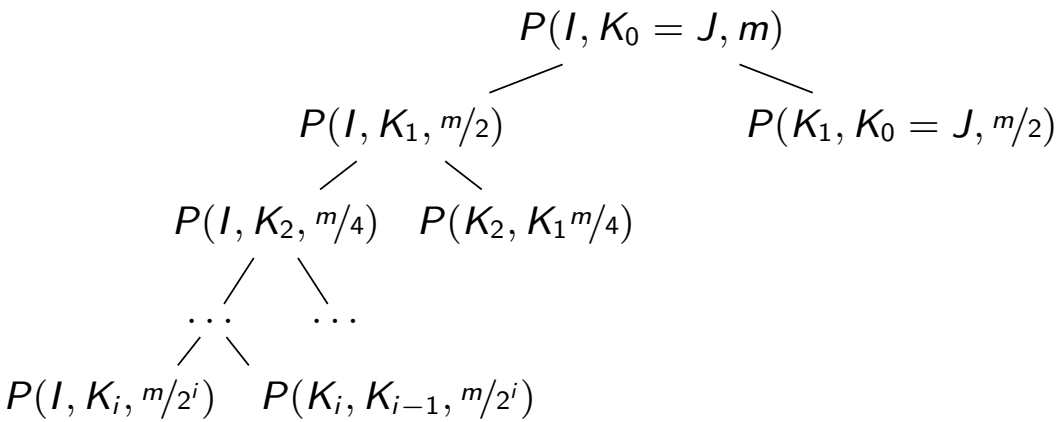

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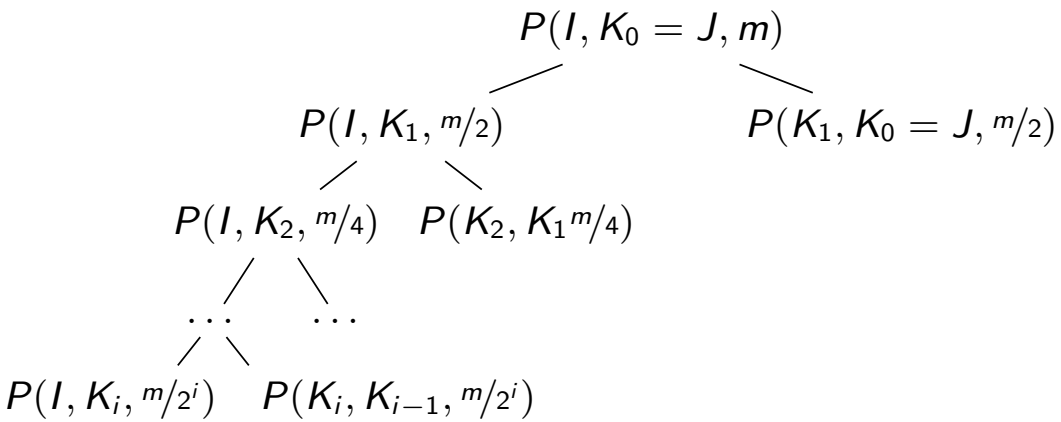

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A. No, we could have just generated all possible (accepting) IDs and try all of them!

PSPACE vs. co-PSPACE

Recap

Recall: A problem is in **co-X** if and only if its complement is in **X**.

Applied to $\mathbf{X} = \mathbf{PSPACE}$: Let $\bar{L} = \Sigma^* \setminus L$

$$\mathbf{DSPACE}(t(n)) = \{L \mid \exists \text{ DTM } M \text{ with } L(M) = L \text{ that decides } L \text{ with } \mathcal{O}(t(n)) \text{ cells } \}$$
$$\mathbf{co\text{-}DSPACE}(t(n)) = \{L \mid \bar{L} \in \mathbf{DSPACE}(t(n))\}$$

$$\mathbf{PSPACE} = \bigcup_{k \in \mathbb{N}} \mathbf{DSPACE}(n^k) \qquad \mathbf{co\text{-}PSPACE} = \bigcup_{k \in \mathbb{N}} \mathbf{co\text{-}DSPACE}(n^k)$$

Also, hardness and completeness is again defined as always, also for **co-PSPACE**.

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NPSPACE Algorithm for $\text{ALL}^c_{\text{NFA}}$ – the complement, which accepts $\langle A \rangle$ if $L(A) \neq \Sigma^*$

Technically, the complement also contains all strings that are not even NFAs. But testing for this is trivial, so no need mentioning this!

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› Since **NPSPACE** = **PSPACE**, we also get $\text{ALL}_{\text{NFA}} \in$ **co-PSPACE**

PSPACE vs. **co-PSPACE**

Theorem w9.1

co-PSPACE = **PSPACE** *(and hence* **co-NPSPACE** = **NPSPACE** = **PSPACE***)*

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Then, show **PSPACE** $\supseteq$ **co-PSPACE** in the analogous way. □

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A lot to unpack here ... Some essential key points to understand:

- Why flipping the result is not allowed in **NP** / **co-NP** proofs,
- but it is allowed in all deterministic classes.
- Thus, **DSPACE** (hence **NSPACE**) and **DTIME** are closed under complementation.

On "Flipping Results", Part 1: Why **NP**/**co-NP** fails.

Disclaimer:

- The following proof is the same as before, but for **NP**/**co-NP**.
- Unless $L \in \mathbf{P}$, or $\mathbf{NP} = \mathbf{co\text{-}NP}$, the following will fail!

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Now, what's wrong about that proof?

On "Flipping Results", Part 1: Why NP/co-NP fails.

Disclaimer:

- The following proof is the same as before, but for **NP**/**co-NP**.
- Unless $L \in \mathbf{P}$, or $\mathbf{NP} = \mathbf{co\text{-}NP}$, the following will fail!

Let $L \in \mathbf{NP}$ and you want to prove $L \in \mathbf{co\text{-}NP}$:

- Let $L \in \mathbf{NP}$. Note that $L \in \mathbf{co\text{-}NP}$ iff $L^c \in \mathbf{NP}$.
- Prove L^c in **NP** via:
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  - Then, flip result for w. This decides L^c, taking poly-time.

Now, what's wrong about that proof?

- To show $L^c \in \mathbf{NP}$, we need o provide an NTM M with $L(M) = L^c$.
- However, our NTM M used in the above proof has $L(M) = L$!
- So, why was that allowed for **PSPACE**?! See next slide!

On "Flipping Results", Part 2: Why **PSPACE**/**co-PSPACE** works.

In our proof:

- Let $L \in$ **PSPACE**. Note that $L \in$ **co-PSPACE** iff $L^c \in$ **PSPACE**.
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- The exact same argument works for all deterministic time classes.

Corollaries

Corollary w9.2

- *All space classes are closed under complementation.*
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- $\text{ALL}_{NFA} \in$ **PSPACE**
- $\text{ALL}^c_{NFA} \in$ **PSPACE**

Thus, to prove membership in space or deterministic time classes, you can choose to decide the complement of the language instead. Pick whatever is easier!

P vs. PSPACE vs. EXPTIME

PSPACE vs. EXPTIME

Theorem w9.1

PSPACE $\subseteq$ EXPTIME

Proof.

› Let $L \in$ **PSPACE**. We will show that any **PSPACE** decider runs in exponential time.

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- How many different TM configurations can we see before running into a loop? (Note that we can't run into a loop! Since M is a decider.)

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  - In total we have at most $c^{p(n)} = \mathcal{O}(|Q| \cdot (n^k) \cdot |\Gamma|^{(n^k)})$ TM configurations.
- So, we can just run M and know that it will not run longer than $c^{p(n)}$, hence in **EXPTIME**

□

P vs. EXPTIME

Intuitively, $\mathbf{P} \subsetneq \mathbf{EXPTIME}$ should hold, since under poly-transformations, we stay withing the respective class, and so have strictly more time available in **EXPTIME**. But, are there really problems that need **EXPTIME**?

To answer this, we require the definition of "small-o" (o), in analogy to "big-O" ($\mathcal{O}$).

Intuitively:

- $f(n) \in \mathcal{O}(g(n))$: f grows at most as fast as g.
- $f(n) \in o(g(n))$: f grows strictly less than g.

Small-o Notation

Definition w9.2

Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$. We say that $f(n) = o(g(n))$ (or $f(n) \in o(g(n))$) if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \ .$$

that is, for any $c > 0$ there exist $n_0 > 0$ such that $f(n) < c \cdot g(n)$, for all $n \geq n_0$.

In comparison:
$f(n) = \mathcal{O}(g(n))$ if there exist $c, n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

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Examples:

- $n \neq o(2n)$ (although $2n$ grows faster than n, but only a constant factor)
- $n = o(\frac{1}{2} n \log n)$ and $n \log n = o(n^2)$

The Time Hierarchy Theorem

Theorem w9.3

If $f : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$, then there exists a decision problem which cannot be solved in worst-case deterministic time $o(f(n))$ but can be solved in worst-case deterministic time $\mathcal{O}(f(n)log(f(n)))$. Thus, $\mathbf{DTIME}(o(f(n))) \subsetneq \mathbf{DTIME}(f(n)log(f(n)))$.

Proof skipped (we only show this for the sake of completeness).

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  Let $f(n) = n^k$. Then,
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  - $L_k \notin \mathbf{DTIME}\big(o(f(n))\big) = \mathbf{DTIME}\big(o(n^k)\big)$.

Back to **P** vs. **EXPTIME**

Corollary w9.4

$\mathbf{P} \subsetneq \mathbf{EXPTIME}$

Proof.

Let $f(n) = 2^n$. By the Time Hierarchy Theorem there is a language L such that

$L \in \mathbf{DTIME}\big(f(n)log(f(n))\big) = \mathbf{DTIME}\big(2^n n\big)$ and $L \notin \mathbf{DTIME}\big(o(f(n))\big) = \mathbf{DTIME}\big(o(2^n)\big)$.

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Since every polynomial n^k satisfies $n^k = o(2^n)$, no poly-time TM decides L. Thus, $L \notin \mathbf{P}$. Also, $n\,2^n = \mathcal{O}\big(2^{cn}\big)$ for some constant $c > 1$, so $L \in \mathbf{DTIME}(2^{cn}) \subseteq \mathbf{EXPTIME}$.

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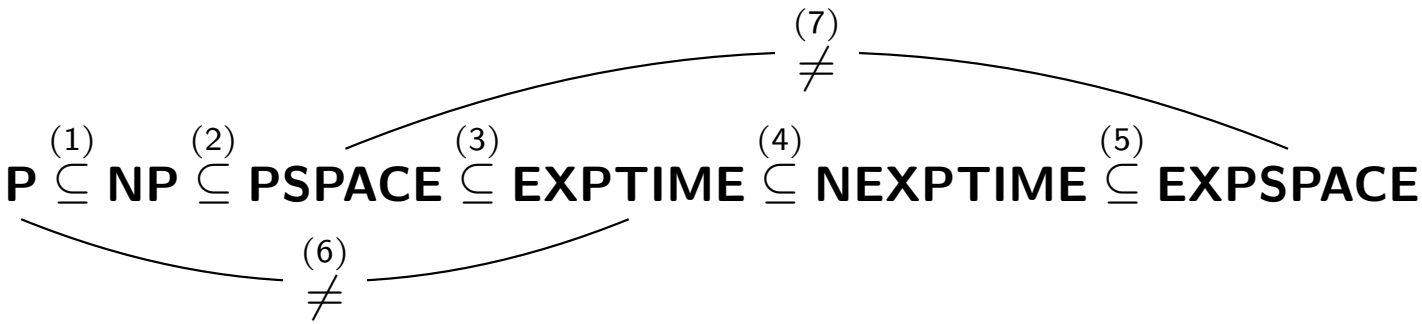
Therefore, $\mathbf{P} \subsetneq \mathbf{EXPTIME}$. □

Relationship Among Complexity Classes: Recap/Summary (Part 1)



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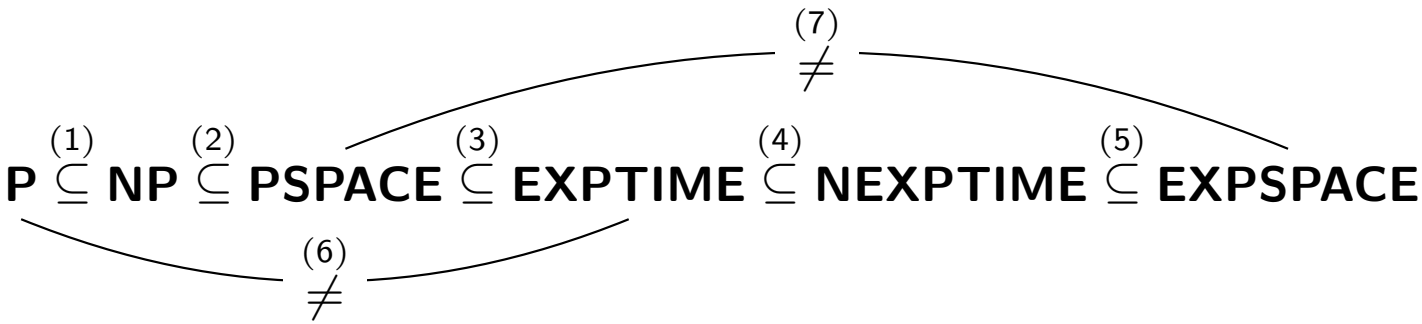

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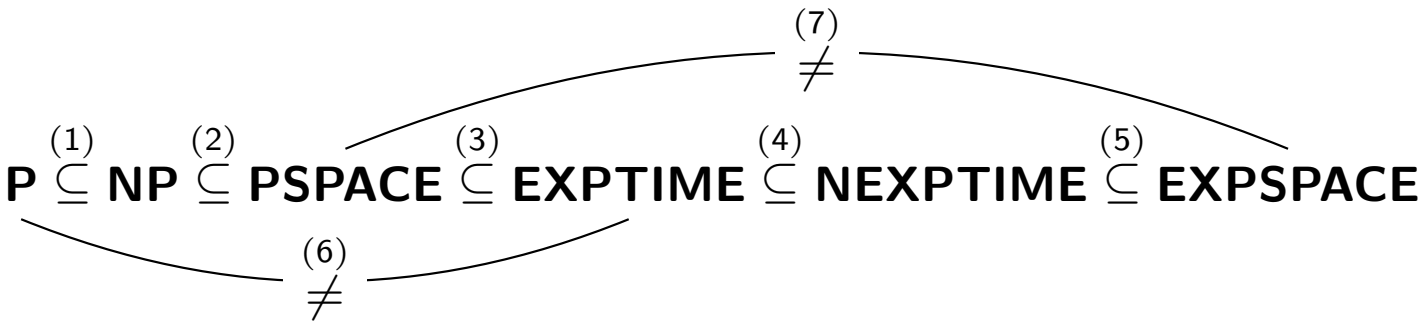

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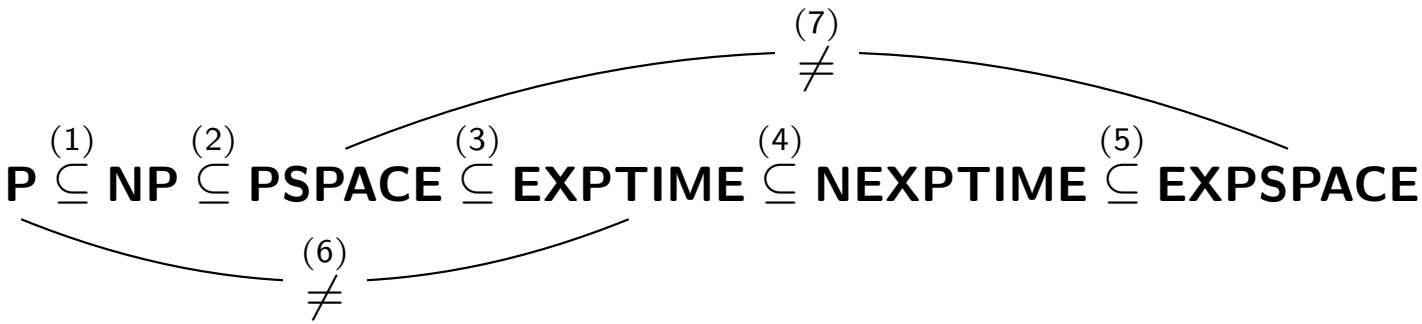

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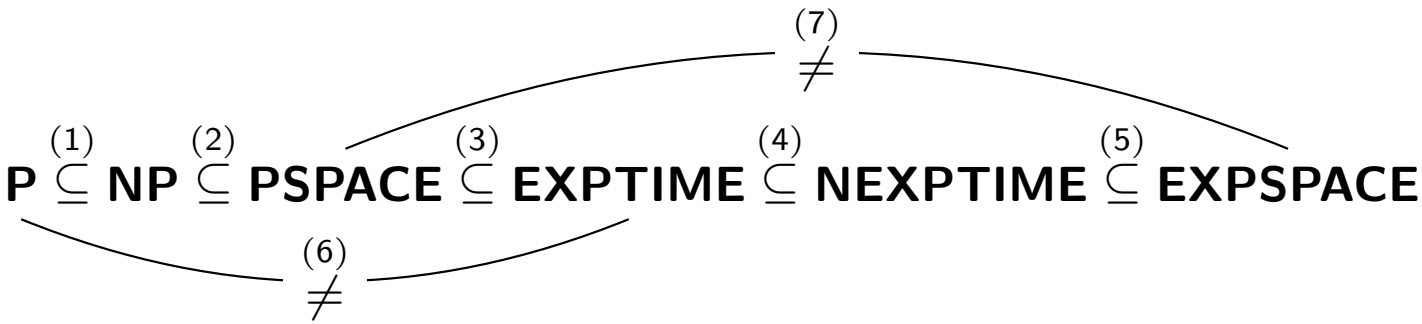

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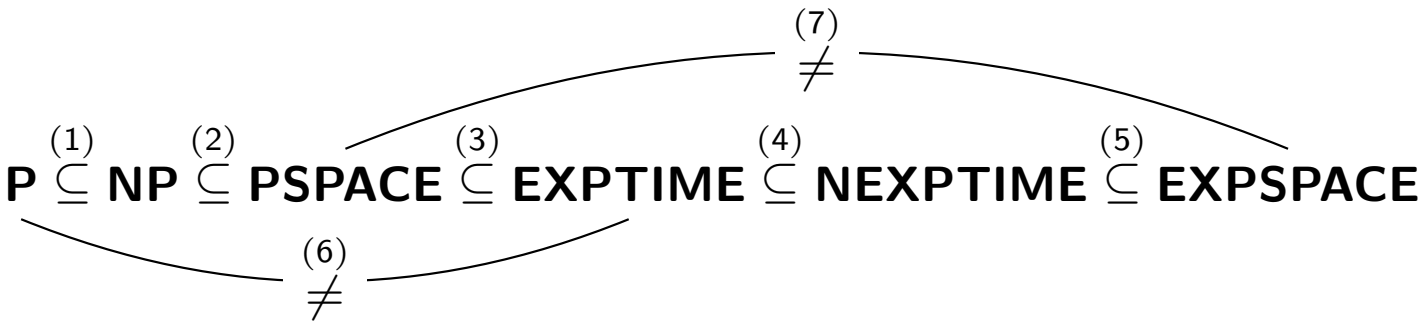

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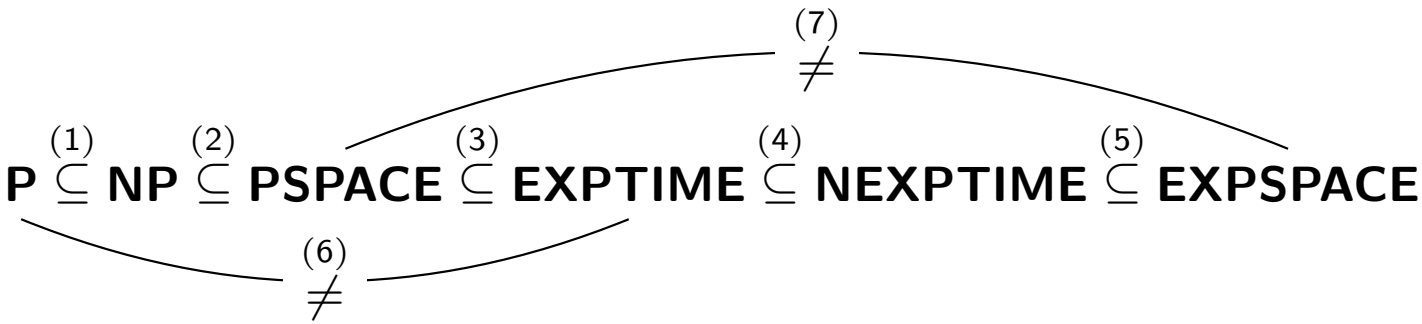

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- (7): Follows from the space hierarchy theorem (not covered).

Relationship Among Complexity Classes: Recap/Summary (Part 2)

Relationships to N- and co-classes:

$$\begin{array}{ccccccccccc}
\textbf{co-P} & \subseteq & \textbf{co-NP} & \subseteq & \textbf{co-PSPACE} & \subseteq & \textbf{co-EXPTIME} & \subseteq & \textbf{co-NEXPTIME} & \subseteq & \textbf{co-EXPSPACE} \\
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Voluntary homework:
Check all additional subset relations for their origin. (I.e., why they hold.)

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What we did not cover; some highlights:

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- Beyond that are even others, such as the **Ackermann** class, sitting above all $k-$**EXPTIME** and $k-$**EXPSPACE** classes.

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- The chain of complexity classes is infinite. E.g., for any $k \geq 1$, there is a class $k-$**EXPTIME** and $k-$**EXPSPACE**.
- Beyond that are even others, such as the **Ackermann** class, sitting above all $k-$**EXPTIME** and $k-$**EXPSPACE** classes.
- There are TM models with different kinds of states (existential and universal), which are convenient for some proofs/problems. They are covered in week 10!

Outlook

What we did not cover; some highlights:

- Between **P** and **NP** sit the **GI**-complete problems (GI = Graph Isomorphy), the question whether two graphs can be renamed to make them isomorphic. Practically extremely relevant!
- Between **NP** and **PSPACE** sits the polynomial hierarchy – an infinite hierarchy of complexity classes. They can be defined via special cases of QBFs. At least Σ_2^p (the next harder class after $\Sigma_1^p = \mathbf{NP}$) is very important for many optimization problems.
- There are probabilistic complexity classes, where TMs have different acceptance criteria (and potentially probabilities).
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- There are TM models with different kinds of states (existential and universal), which are convenient for some proofs/problems. They are covered in week 10!
- **And certainly many more ...** Check out complexityzoo.net – 550 classes so far!)