Intractability of Optimal Multi-Agent Pathfinding on Directed Graphs

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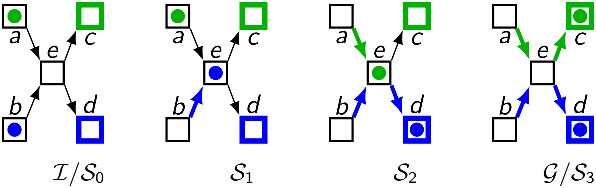
A Directed Multi-Agent Path-Finding (diMAPF) Example

- Initially at $\mathcal{I} \equiv \mathcal{S}_0$, the green agent A_g is at vertex a, and the blue agent A_b is at b. The goal for A_g is c, for A_b is d. In between $\mathcal{I}$ and $\mathcal{G}$, A_g waits at vertex a for A_b to pass vertex e first.



diMAPF: Outline

- Definitions
- Decisional diMAPF
- Optimal diMAPF
- Summary

Definitions

- Let $\mathcal{A}$ be a finite set of agents, and $G = \langle V, E \rangle$ a directed graph, where V is a finite set of vertices and $E \subseteq V \times V$ is a finite set of directed edges.
- An agent in $\mathcal{A}$ can move from $v_i \in V$ to $v_j \in V$ if $(v_i, v_j) \in E$ is an edge in the directed graph G.
- A state $\mathcal{S}$ defines a distribution of all agents from $\mathcal{A}$, in vertices from V.
- Time is measured in steps. A step σ defines a step-wise movement of all agents, which changes a state $\mathcal{S}$ into its successor $\mathcal{S}^{succ}$.
- It is required that the movement of all agents in σ, between $\mathcal{S}$ and $\mathcal{S}^{succ}$, should be applicable ones, and the applicability of agent movements is defined by the principles of precondition and frame axioms (in classical AI planning).

Definitions (Cont'd)

MAPF A four-tuple $\langle G, \mathcal{A}, \mathcal{I}, \mathcal{G} \rangle$, where G is an undirected graph, $\mathcal{A}$ is a set of agents, $\mathcal{I}$ is the initial situation, and $\mathcal{G}$ is the goal situation. Is there a sequence $\vec{\mathcal{S}}$ that moves agents in $\mathcal{A}$ from $\mathcal{I}$ to $\mathcal{G}$?

diMAPF$^{\mathcal{R}}$ A directed MAPF problem on a set of restrictions $\mathcal{R}$, e.g.,:

- **d≤3** Let "$dgr(i,j)$" denote a vertex in a digraph having in-degree i and out-degree j. For any vertex v in the graph, if $dgr(i,j)$ holds for v, then $i + j \leq 3$ and $i, j \leq 2$ hold as well.
- **dag** G is acyclic, i.e., a DAG.
- **pl** G is planar. That is, it can be drawn on the plane in a way such that none of its edges intersect with each other.
- **sc** G is strongly connected, i.e., every pair of vertices u and v should have a path in each direction between them.
- **uc** G is unilaterally connected, i.e., every pair of vertices u and v in G should have a path in at least one direction between them.
- **wc** G is weakly connected, i.e., if there is a path between every pair of vertices u and v in the underlying undirected graph.

Intractability of Decisional diMAPF Problems

Theorem 1. The problem *diMAPF* is NP-hard, and in PSPACE; NP-complete, if $\mathcal{G}$ is a directed acyclic graph (**dag**); NP-complete, if $\mathcal{G}$ is a strongly connected digraph (**sc**). (Thm 1, Props 2/3, and Thm 4 in (Nebel 2020[1]), Thms 18/19 in (Nebel 2023[2]))

Theorem 2. A Rectilinear-Planar Monotone Sided 3SAT (*RPMS-3SAT*) instance is a Boolean formula in the 3SAT format, rectilinearly-planar, and monotone. In addition, the clauses are sided: All positive/negative clauses are on the side above/below the variable line, respectively. RPMS-3SAT is NP-complete. (Thm 1 in Mark de Berg and et. al. 2010[3])

Theorem 3. $\mathit{diMAPF}^{\mathit{dag},\mathit{pl}}$ is NP-complete.

Corollary 4. $\mathit{diMAPF}^{\mathit{dag},\mathit{pl},\mathrm{d}\leq 3}$ is NP-complete.

Theorem 5. $\mathit{diMAPF}^{\mathit{dag},\mathit{uc}}$ is NP-complete.

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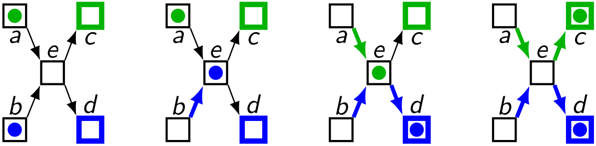
[1] Bernhard Nebel, "On the computational complexity of multi-agent pathfinding on directed graphs", in Proc. of the 30th ICAPS, pp. 212–216, (2020).

[2] Bernhard Nebel, "The small solution hypothesis for MAPF on strongly connected directed graphs is true", in Proc. of the 33rd ICAPS, (2023).

[3] Mark de Berg and Amirali Khosravi, "Optimal binary space partitions in the plane", in Computing and Combinatorics, pp. 216–225, Berlin, Heidelberg, 2010.

Optimal diMAPF (OdP) Examples

- Initially, A_g is at a, and A_b is at b. The goal for A_g is c, for A_b is d.



$\mathcal{I}/\mathcal{S}_0$ $\mathcal{S}_1$ $\mathcal{S}_2$ $\mathcal{G}/\mathcal{S}_3$

- Path lengths for both agents are 2: $|p_g| = 2$ and $|p_b| = 2$, and $\mathcal{P} = \{p_g, p_b\}$, containing all paths for all agents.
- The maximal distance of the example $\underset{p \in \mathcal{P}}{\operatorname{argmax}}(|p|)$, is 2.
- Total distance is $2 + 2 = 4$.
- Total number of steps from $\mathcal{I}$ to $\mathcal{G}$ (i.e., makespan) is $|\Sigma| = 3$.
- The minimum **max**imal distance of the example is actually 2.
- The minimum **tot**al distance is $2 + 2 = 4$.
- The minimum **m**ake**s**pan is actually 3.

OdP Definitions

Definition 6 (diMAPF$^{\mathcal{R}}_{\langle O,\mathbf{B}\rangle}$). *diMAPF*$^{\mathcal{R}}_{\langle O,B\rangle}$ is a *diMAPF* problem subject to restrictions $\mathcal{R}$ and an optimization criterion of type O, which is bounded by $B \in \mathbb{Z}^+$.
To decide *diMAPF*$^{\mathcal{R}}_{\langle O,B\rangle}$ with $O \in \{ms, max, tot\}$ means to answer: Is there a $\vec{\mathcal{S}}$, such that

- If $O = ms$, then: $|\Sigma| \leq B$?
- If $O = max$, then: $\underset{p \in \mathcal{P}}{\operatorname{argmax}}\big(|p| \leq B\big)$?
- If $O = tot$, then: $\left(\sum_{p \in \mathcal{P}} |p|\right) \leq B$?

Complexity of OdP With Makespan Restrictions

Definition 7. Given a $diMAPF^{\mathcal{R}}_{\langle ms, lb_{ms}\rangle}$ problem, lb_{ms} is the lower bound makespan, if there does not exist a $\vec{\mathcal{S}}$, which is a solution to $diMAPF^{\mathcal{R}}$, and $|\Sigma| < lb_{ms}$.

Definition 8 ($\mathbf{LB_{ms}}$ of $\mathbf{diMAPF^{\mathcal{R}}}$). Given a $diMAPF^{\mathcal{R}}$ problem, and a set $\mathcal{P}_{min}$ containing the shortest paths from their initial vertices to their goal vertices, for all the agents in the problem. Suppose $\hat{p} \in \mathcal{P}_{min}$ is a path whose length is the maximal one among all paths, $LB_{ms} = |\hat{p}|$.

Proposition 1. Given a $diMAPF^{\mathcal{R}}$, $\mathcal{P}_{min}$, thus LB_{ms}, can be calculated in poly-time (Dijkstra's Algorithm to find shortest paths).

Theorem 9. $diMAPF^{dag,pl}_{\langle ms, LB_{ms}\rangle}$ is NP-complete.

OdP with Restrictions on Path Length

Definition 10. We first identify respectively in the following two lower bounds for maximal individual travel distance, and for total distance.

- **(lower bound of maximal distance: $\mathbf{LB_{max}}$)** We can again find out $\mathcal{P}_{min}$. Among them, we pick up $\hat{p}$ the longest one, whose length $|\hat{p}| = LB_{max}$, which serves as the lower bound for B (justification is similar to the one for lb_{ms}/LB_{ms}).
- **(lower bound of total distance: $\mathbf{LB_{tot}}$)** The actual LB_{tot} should be instantiated into $LB_{tot} = \sum_{p \in \mathcal{P}_{min}} |p|$. Any path $p \in \mathcal{P}_{min}$ is the shortest one for its agent, the sum of all path lengths in any solution to the given problem thus can not be shorter than LB_{tot}.

Theorem 11. $diMAPF^{dag,pl}_{\langle max, LB_{max} \rangle}$ is NP-complete.

Theorem 12. $diMAPF^{dag,pl}_{\langle tot, LB_{tot} \rangle}$ is NP-complete.

Theorem 13. All three of the following problems are NP-complete: $diMAPF^{dag,pl,\mathrm{d}\leq 3}_{\langle ms, LB_{ms} \rangle}$, $diMAPF^{dag,pl,\mathrm{d}\leq 3}_{\langle max, LB_{max} \rangle}$ and $diMAPF^{dag,pl,\mathrm{d}\leq 3}_{\langle tot, LB_{tot} \rangle}$.

Conclusions

This research investigates the complexity of decisional/optimal *diMAPF* problems.

- the *diMAPF* problems are subject to the optimization criterion of *ms*, *max*, and *tot*; Meanwhile,
- a variety of different special cases of the problem are considered. For example, the digraphs are restricted to be planar, or/and are with bounded vertex-degrees.
- Surprisingly, even if it is applied with the smallest-possible value, optimal *diMAPF* remains to be NP-hard, for the criteria of *ms*, *max*, and *tot*, respectively.
- These complexity results have important real-world implications. For instance, we now know that when on a two-dimensional space only (**pl**), with one-way roads only (**dag**), and at most three-branch junctions only ($\mathbf{d}\leq\mathbf{3}$), MAPF is already computationally challenging.

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